

LAGUERRE AND MEIXNER ORTHOGONAL BASES IN THE ALGEBRA OF SYMMETRIC FUNCTIONS

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ABSTRACT. Analogs of Laguerre and Meixner orthogonal polynomials in the algebra of symmetric functions are studied. The work is motivated by a connection with a model of infinite-dimensional Markov dynamics.

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1. INTRODUCTION

1.1. **Preface.** Let Sym denote the graded algebra of symmetric functions. The theory of symmetric functions deals with various homogeneous bases in Sym . A simple yet fundamental example is the basis of Schur symmetric functions. More sophisticated examples — Hall-Littlewood, Jack, and Macdonald symmetric functions (which are on the top of the hierarchy) — form a one- or two-parameter deformation of the Schur functions [Ma95]. Each of these bases is an orthogonal basis with respect to an appropriate inner product in Sym .

The aim of the paper is to introduce two new families of orthogonal bases in Sym : We call them the *Laguerre* and *Meixner* symmetric functions. In contrast to the symmetric functions mentioned above, the Laguerre and Meixner symmetric functions are *inhomogeneous* elements of Sym . As their names suggest, they are somewhat related to the Laguerre and Meixner orthogonal polynomials.

Natural multivariate analogs of classical orthogonal polynomials have been investigated in the widely cited but unpublished manuscript by Macdonald [Ma87] and in a cycle of notes by Lassalle [La91a], [La91b], [La91c]. However, the Laguerre and Meixner symmetric functions cannot be obtained from the corresponding N -variate symmetric polynomials simply by letting $N \rightarrow \infty$, like as the Schur symmetric functions arise from the N -variate symmetric Schur polynomials.

The construction of the present paper is based on the following trick: we treat N , the number of variables, as an independent parameter and then perform analytic continuation into complex domain with respect to this parameter. As a result, the Laguerre symmetric functions depend on two parameters while the Laguerre polynomials involve a single parameter only, and the Meixner symmetric functions acquire three parameters instead of the conventional two parameters.

Another feature of the construction is that the most natural realization of the Laguerre and Meixner symmetric functions is achieved when Sym is realized as the algebra of *supersymmetric functions*.

1.2. **Description of results.** 1) As usual in Sym , the basis elements are indexed by arbitrary partitions ν . The Laguerre symmetric functions \mathfrak{L}_ν are defined in terms of their expansion in the basis $\{S_\nu\}$ of Schur functions, see (4.3). A similar expansion for the Meixner symmetric functions \mathfrak{M}_ν is (4.12), but here the Schur functions

are replaced by their factorial analogs, the so-called Frobenius-Schur functions FS_ν . These expansions may be viewed as analogs of the well-known explicit formulas for the univariate Laguerre and Meixner polynomials.

2) Alternatively, the elements \mathfrak{L}_ν and \mathfrak{M}_ν can be characterized by two properties:

- First,

$$\mathfrak{L}_\nu = S_\nu + \text{lower degree terms}, \quad \mathfrak{M}_\nu = S_\nu + \text{lower degree terms}.$$

- Second, \mathfrak{L}_ν and \mathfrak{M}_ν are eigenfunctions of some operators in Sym , the Laguerre operator \mathfrak{D}^L and the Meixner operator \mathfrak{D}^M , respectively:

$$\mathfrak{D}^L \mathfrak{L}_\nu = -|\nu| \mathfrak{L}_\nu, \quad \mathfrak{D}^M \mathfrak{M}_\nu = -|\nu| \mathfrak{M}_\nu.$$

This characterization is similar to the well-known characterization of classical orthogonal polynomials as polynomial eigenfunctions of suitable differential or difference operators of hypergeometric type.

3) Operators \mathfrak{D}^L and \mathfrak{D}^M are initially defined by their action on the Schur and Frobenius-Schur functions, see (4.7) and (4.13). Alternatively, the Laguerre operator \mathfrak{D}^L can be written as a second order differential operator in formal variables e_1, e_2, \dots (the elementary symmetric functions) or h_1, h_2, \dots (the complete homogeneous symmetric functions), see Theorem 4.10 and Corollary 4.13. As for Meixner operator \mathfrak{D}^M , it can be written as a kind of difference operator on the Young graph \mathbb{Y} , see Proposition 4.25.

4) We define inner products in Sym in which the functions \mathfrak{L}_ν and \mathfrak{M}_ν are orthogonal. This is done in terms of formal moment functionals φ^L and φ^M on the space Sym , which are explicitly computed in Proposition 5.2. We also get explicit expressions for the squared norms of \mathfrak{L}_ν and \mathfrak{M}_ν , see Theorem 5.4.

5) To achieve a full analogy with orthogonal polynomials we have to show that the formal moment functionals can be written as the expectations under some probability measures, in an appropriate functional realization of the algebra Sym . Such measures (we call them the *orthogonality measures*) are exhibited in Theorem 5.18 and Theorem 5.27.

6) Finally, as is well known, the Meixner polynomials are discrete counterparts of the Laguerre polynomials: the latter are limits of the former. Likewise, the Laguerre differential operator is a scaling limit of the Meixner difference operator, and the continuous gamma distribution (the weight measure for the Laguerre polynomials) can be obtained through a scaling limit from the discrete negative binomial distribution (the weight for the Meixner polynomials). We show that similar limit relations hold for the Laguerre and Meixner symmetric functions. In particular, Theorem 5.28 explains how the Laguerre orthogonality measure can be approximated by the Meixner orthogonality measures.

7) In Appendix, we briefly describe a degeneration of the Meixner symmetric functions leading to one more family of symmetric functions, which we call the Charlier functions. The corresponding orthogonal measure is the well-known poissonized Plancherel measure.

Note that the Laguerre and Meixner orthogonality measures have a representation-theoretic origin. They were earlier studied in a cycle of papers by Borodin and the author, see [BO00a], [BO00b], [BO06a], [BO06c], [BO09], [Ol03a], [Ol03b]. The Meixner orthogonality measures are the so-called *z-measures* on \mathbb{Y} . The Laguerre orthogonality measures arise in noncommutative harmonic analysis on the infinite symmetric group: they govern the spectral decomposition of some unitary representations. The Laguerre orthogonality measures live on an infinite-dimensional cone $\tilde{\Omega}$ whose base is Thoma's simplex Ω (a kind of dual space to the infinite symmetric group).

Note also that both the Laguerre and Meixner orthogonality measures give rise to determinantal point processes, see [BO00a], [BO06a], [BO06c].

Next, as is shown in [BO06a], the Meixner difference operator on the Young graph \mathbb{Y} serves as the infinitesimal generator of a jump Markov process. A similar (but more difficult result) holds in the Laguerre case, too: the Laguerre operator generates a diffusion process on the cone $\tilde{\Omega}$, see [Ol10b]. The fact that the Laguerre and Meixner symmetric functions provide an explicit diagonalization of these Markov generators was the main motivation of the present work.

Finally, the Charlier operator (6.1), which is a degeneration of the Meixner operator, serves as the infinitesimal generator in a model of Markov dynamics related to the Plancherel measure, see [BO06b].

1.3. Notes. The present paper is a detailed exposition of part of the results announced in [Ol10b]. Other results of [Ol10b], which concern the “Laguerre” Markov dynamics on the Thoma cone, are the subject of a separate detailed paper under preparation.

In the study of the *z-measures*, the trick of analytic continuation in N was exploited in [BO06a] and [BO06c]; in that papers we developed some observations made earlier in [BO00a, §4 and Remark 5.5].

In the context of symmetric functions, analytic continuation in N was earlier used by Rains [Ra05] and then by Sergeev and Veselov [SV09a], [SV09b], [SV09b] to lift BC_N -type symmetric polynomials to infinite dimensions. Rains' formidable paper deals with the Koornwinder polynomials (a multivariate version of the Askey-Wilson polynomials), and Sergeev and Veselov deal with the Jacobi polynomials.

The approach of Sergeev and Veselov was extended to the case of Hermite and Laguerre symmetric functions by Desrosiers and Hallnäs [DH11]. It is interesting to compare their results and those of the present paper. There are some intersections in what concerns the Laguerre case; the construction of [DH11] is more general as it involves the extra (Jack) parameter $\alpha = 1/\theta$; [DH11] also contains a number of

nice additional results about the Laguerre symmetric functions. On the other hand, Desrosiers and Hallnäs do not discuss the Meixner case and the topics related to orthogonality.

Note that one can extend the construction of the Meixner symmetric functions to the case of the Jack parameter $\theta = 1/\alpha$. However, in my understanding, these more general Meixner functions no longer live in Sym , but should be defined as elements of a certain algebra of functions on \mathbb{Y} (the algebra \mathbb{A}_θ of θ -regular functions on \mathbb{Y} , in the terminology of [Ol10a]; it is isomorphic to the algebra of θ -shifted symmetric functions from [OO97b]). In the special case $\theta = 1$, there is a natural isomorphism between this algebra and algebra Sym (see [ORV03]) but it seems that there is no distinguished way to identify \mathbb{A}_θ with Sym when $\theta \neq 1$.

The importance of the supersymmetric realization of the algebra Sym for the representation theory of the infinite symmetric group became clear after the work of Vershik and Kerov [VK81], [VK90]. For subsequent developments of their ideas, see [KOO98], [ORV03], [IO03]; see also numerous papers by Sergeev and Veselov on integrable systems related to Lie superalgebras.

2. CLASSICAL LAGUERRE AND MEIXNER POLYNOMIALS

Here we collect a few necessary formulas concerning the classical univariate Laguerre and Meixner polynomials. All these formulas are easily extracted from [KS96]. The reader should keep in mind that our normalization of these polynomials and our notation differ from the conventional ones, as we prefer to work with *monic* polynomials.

The *classical Laguerre polynomials* $L_n(x)$ depend on a parameter $b > 0$; they form an orthogonal system on the half-line $\mathbb{R}_+ := \{x \in \mathbb{R} : x \geq 0\}$ with respect to the weight measure

$$\gamma_b(dx) = \frac{1}{\Gamma(b)} x^{b-1} e^{-x} dx, \quad x \in \mathbb{R}_+, \quad (2.1)$$

which is the probability *gamma distribution*.

The second order differential operator

$$D^\mathbb{L} = x \frac{d^2}{dx^2} + (b - x) \frac{d}{dx},$$

is formally symmetric with respect to γ_b , and the Laguerre polynomials are eigenfunctions of $D^\mathbb{L}$:

$$D^\mathbb{L} L_n = -n L_n.$$

Moreover, they are the only polynomial eigenfunctions of $D^\mathbb{L}$. Note a useful formula for the action of $D^\mathbb{L}$ on the monomials:

$$D^\mathbb{L} x^n = -n x^n + n(n + b - 1) x^{n-1}. \quad (2.2)$$

Introduce a notation for the falling factorial powers of a variable:

$$x^{\downarrow m} = x(x - 1) \dots (x - m + 1) = (-1)^m (-x)_m, \quad m = 0, 1, \dots$$

with the understanding that $x^{\downarrow 0} = 1$. Here $(x)_m = x(x+1)\dots(x+m-1)$ is the standard notation for the raising factorial power, aka the *Pochhammer symbol*. In this notation, the monic Laguerre polynomials can be explicitly written as

$$L_n(x) = (-1)^n (b)_n {}_1F_1(-n; b; x) = (b)_n \sum_{m=0}^n (-1)^{m-n} \frac{n^{\downarrow m}}{(b)_m m!} x^m, \quad (2.3)$$

where ${}_1F_1$ is the confluent hypergeometric series.

An important constant is the squared norm with respect to the weight measure,

$$(L_n, L_n) := \int_0^{+\infty} (L_n(x))^2 \gamma_b(dx) = (b)_n n!. \quad (2.4)$$

The *classical Meixner polynomials* $M_n(x)$ depend on the same parameter $b > 0$ and an additional parameter $\xi \in (0, 1)$. They are orthogonal with respect to the *negative binomial distribution* on the half-lattice $\mathbb{Z}_+ := \{x \in \mathbb{Z} : x \geq 0\}$ given by

$$\eta_{b,\xi} = (1-\xi)^b \sum_{x \in \mathbb{Z}_+} \frac{(b)_x}{x!} \xi^x \delta_x,$$

where δ_x denotes the Dirac measure at x .

One associates with the family $\{M_n\}$ the second order difference operator on \mathbb{Z}_+ defined by

$$\begin{aligned} D^M f(x) &= \frac{\xi(b+x)}{1-\xi} f(x+1) + \frac{x}{1-\xi} f(x-1) \\ &\quad - \frac{\xi(b+x) + x}{1-\xi} f(x). \end{aligned}$$

It is formally symmetric with respect to the weight function $\eta_{b,\xi}$ and annihilates the constants. The Meixner polynomials can be characterized as the only polynomial eigenfunctions of this operator:

$$D^M M_n = -n M_n.$$

The operator D^M admits a nice expression in the inhomogeneous basis $\{x^{\downarrow n}\}$:

$$D^M x^{\downarrow n} = -n x^n + \frac{\xi}{1-\xi} n(n+b-1) x^{\downarrow(n-1)}. \quad (2.5)$$

Here is an explicit expression of the monic Meixner polynomials:

$$\begin{aligned} M_n(x) &= \left(\frac{\xi}{\xi-1} \right)^n (b)_n {}_2F_1 \left(-n, -x; b; \frac{\xi-1}{\xi} \right) \\ &= (b)_n \sum_{m=0}^n \left(\frac{\xi}{\xi-1} \right)^{n-m} \frac{n^{\downarrow m}}{(b)_m m!} x^{\downarrow m}, \end{aligned} \quad (2.6)$$

where ${}_2F_1$ is the Gauss hypergeometric series.

The squared norm with respect to the weight $\eta_{b,\xi}$ on \mathbb{Z}_+ is given by

$$(M_n, M_n) = \xi^n (1 - \xi)^{-2n} (b)_n n!. \quad (2.7)$$

Proposition 2.1. *In a scaling limit transition, as $\xi \rightarrow 1$ and simultaneously the lattice is shrunk with small coefficient $1 - \xi$, the Meixner polynomials converge to the Laguerre polynomials:*

$$\lim_{\xi \rightarrow 1} (1 - \xi)^n M_n((1 - \xi)^{-1}x) = L_n(x).$$

Note that the multiplication by $(1 - \xi)^n$ is needed to keep the top degree coefficient of the polynomial to be equal to 1.

Proof. Compare (2.6) and (2.3). Set $\varepsilon = 1 - \xi$ and

$$x^{\downarrow m, \varepsilon} = x(x - \varepsilon) \dots (x - (m - 1)\varepsilon).$$

Substituting $x \rightarrow (1 - \xi)^{-1}x$ in (2.6) and multiplying by $(1 - \xi)^n$ results in replacing $x^{\downarrow m}$ by $x^{\downarrow m, \varepsilon}$. It follows that the desired convergence holds term-wise, for every fixed $m = 0, \dots, n$. \square

Remark 2.2. (i) The above asymptotic relation is explained by the fact that in the same scaling limit regime, the negative binomial distribution approximates the gamma distribution.

(ii) An alternative explanation can be extracted from the comparison of (2.2) and (2.5). As above, make the change of a variable $x \rightarrow \varepsilon^{-1}x$, which means that the lattice \mathbb{Z}_+ is shrunk with small scale factor ε . The analog of (2.5) for the rescaled difference operator $D^{\mathbf{M}, \varepsilon}$ is

$$D^{\mathbf{M}, \varepsilon} x^{\downarrow n, \varepsilon} = -n x^{\downarrow n, \varepsilon} + \xi n(n + b - 1) x^{\downarrow (n-1), \varepsilon}.$$

In the limit as $\varepsilon \rightarrow 0$ this leads to (2.2).

3. THE N -VARIATE LAGUERRE AND MEIXNER SYMMETRIC POLYNOMIALS

3.1. From univariate polynomials to N -variate symmetric polynomials.

Fix $N = 1, 2, \dots$ and denote by $\text{Sym}(N)$ the subalgebra of symmetric polynomials in $\mathbb{R}[x_1, \dots, x_N]$. Let $\mathbb{Y}(N)$ denote the set of all integer partitions with at most N nonzero summands; we write such partitions as N -dimensional vectors

$$\nu = (\nu_1, \dots, \nu_N) \in \mathbb{Z}_+^N, \quad \nu_1 \geq \dots \geq \nu_N,$$

and identify them with Young diagrams with at most N nonzero rows. The quantity $|\nu| := \nu_1 + \dots + \nu_N$ is equal to the number of boxes in ν . By \emptyset we denote the empty Young diagram represented by the zero vector $(0, \dots, 0)$, and V_N is our shorthand notation for the Vandermonde:

$$V_N = V_N(x_1, \dots, x_N) := \prod_{1 \leq i < j \leq N} (x_i - x_j).$$

Let $\{\phi_n\}_{n=0,1,\dots}$ be an arbitrary basis in $\mathbb{R}[x]$ formed by monic polynomials with $\deg \phi_n = n$. There is a well-known way to construct from $\{\phi_n\}$ a basis $\{\phi_{\nu|N}\}$ in $\text{Sym}(N)$ labelled by Young diagrams $\nu \in \mathbb{Y}(N)$. Namely, for any $\nu \in \mathbb{Y}(N)$,

$$\phi_{\nu|N}(x_1, \dots, x_N) := \frac{\det[\phi_{\nu_i+N-i}(x_j)]}{V_N(x_1, \dots, x_N)} \quad (3.1)$$

is a symmetric polynomial, because the $N \times N$ determinant in the numerator is an antisymmetric polynomial and hence is divisible by V_N . As ν ranges over $\mathbb{Y}(N)$, we get the desired basis in $\text{Sym}(N)$. Note that

$$\deg \phi_{\nu|N} = |\nu|.$$

We will employ this construction in four particular cases:

- Setting $\phi_n = x^n$ produces the *Schur polynomials*, which we denote by $S_{\nu|N}$.
- Setting

$$\phi_n = x^{\downarrow n} := x(x-1)\dots(x-n+1)$$

produces the so-called *factorial Schur polynomials* denoted as $S_{\nu|N}^f$.

- Setting $\phi_n = L_n$ produces the *N-variate Laguerre polynomials* denoted as $L_{\nu|N}$ or, in more detail, $L_{\nu|N,b}$.

- Setting $\phi_n = M_n$ produces the *N-variate Meixner polynomials* denoted as $M_{\nu|N} = M_{\nu|N,b,\xi}$.

The Schur polynomials are homogeneous, the other polynomials $\phi_{\nu|N}$ are not, but the top degree homogeneous component in $\phi_{\nu|N}$ always coincides with $S_{\nu|N}$:

$$\phi_{\nu|N} = S_{\nu|N} + \text{lower degree terms.}$$

It follows that every family of the form $\{\phi_{\nu|N}\}$ is a basis in $\text{Sym}(N)$. Moreover, any family $\{\phi_{\nu|N}\}$ is consistent with the canonical filtration in $\text{Sym}(N)$ in the sense that, for any $n = 0, 1, 2, \dots$, the set of the basis elements of degree $\leq n$ form a basis in the space of symmetric polynomials of degree $\leq n$.

3.2. Expansions. Assume $\{\phi'_n\}$ is another basis in $\mathbb{R}[x]$ formed by monic polynomials, and let $t(n, m)$ denote the transition coefficients between the two bases,

$$\phi_n = \sum_{m=0}^n t(n, m) \phi'_m.$$

Proposition 3.1. *Let $\{\phi_n\}$, $\{\phi'_n\}$, and $t(n, m)$ be as above. For any $\nu \in \mathbb{Y}(N)$ we have the expansion*

$$\phi_{\nu|N} = \sum_{\mu: \mu \subseteq \nu} t(\nu, \mu) \phi'_{\mu|N},$$

where summation is taken over all diagrams μ contained in ν and the coefficients are $N \times N$ determinants built from the numbers $t(n, m)$,

$$t(\nu, \mu) := \det[t(n_i, m_j)], \quad n_i = \nu_i + N - i, \quad m_j = \mu_j + N - j.$$

Proof. Direct verification. \square

For $\nu \in \mathbb{Y}(N)$ and $\mu \subseteq \nu$, denote by $\dim \nu / \mu$ the number of standard tableaux of skew shape ν / μ .

Corollary 3.2. *The following expansions hold*

$$L_{\nu|N,b} = \sum_{\mu: \mu \subseteq \nu} (-1)^{|\nu| - |\mu|} \frac{\dim \nu / \mu}{(|\nu| - |\mu|)!} \prod_{i=1}^N \frac{(\nu_i + N - i)!(b)_{\nu_i + N - i}}{(\mu_i + N - i)!(b)_{\mu_i + N - i}} \cdot S_{\mu|N} \quad (3.2)$$

and

$$\begin{aligned} M_{\nu|N,b,\xi} &= \sum_{\mu: \mu \subseteq \nu} (-1)^{|\nu| - |\mu|} \left(\frac{\xi}{1 - \xi} \right)^{|\nu| - |\mu|} \frac{\dim \nu / \mu}{(|\nu| - |\mu|)!} \\ &\quad \times \prod_{i=1}^N \frac{(\nu_i + N - i)!(b)_{\nu_i + N - i}}{(\mu_i + N - i)!(b)_{\mu_i + N - i}} \cdot S_{\mu|N}^f \end{aligned} \quad (3.3)$$

Proof. Apply Proposition 3.1 for $\phi_n = L_n$ and $\phi'_n = x^n$. By (2.3)

$$t(n, m) = (-1)^{n-m} \frac{(b)_n n^{\downarrow m}}{(b)_m m!}.$$

Then we get, setting $n_i = \nu_i + N - i$ and $m_j = \mu_j + N - j$,

$$\begin{aligned} t(\nu, \mu) &= (-1)^{|\nu| - |\mu|} \prod_{i=1}^n \frac{(b)_{n_i}}{(b)_{m_i} m_i!} \cdot \det[n_i^{\downarrow m_j}] \\ &= (-1)^{|\nu| - |\mu|} \prod_{i=1}^n \frac{n_i! (b)_{n_i}}{m_i! (b)_{m_i}} \cdot \frac{\det[n_i^{\downarrow m_j}]}{\prod n_i!} \end{aligned}$$

On the other hand,

$$\frac{\det[n_i^{\downarrow m_j}]}{\prod n_i!} = \det \left[\frac{1}{(n_i - m_j)!} \right] = \frac{\dim \nu / \mu}{(|\nu| - |\mu|)!},$$

see, e.g., the proof of Proposition 1.2 in [ORV03] (in that proof there is a minor typo: the correct reference to Macdonald's book should be [Ma95, §I.7, Ex. 6]). This gives (3.2).

The proof of (3.3) is just the same: we take $\phi_n = M_n$ and $\phi'_n = x^{\downarrow n}$, and use (2.5). \square

3.3. Differential/difference operators. Let $\{\phi_n\}$ be as above. Consider the linear operator $D: \mathbb{R}[x] \rightarrow \mathbb{R}[x]$ defined by $D\phi_n = -n\phi_n$ for $n = 0, 1, 2, \dots$.

Proposition 3.3. *Fix $N = 1, 2, \dots$ and denote by $D^{(i)}$ a copy of D acting on the i th variable x_i , $1 \leq i \leq N$. The formula*

$$D_N = V_N^{-1} \circ (D^{(1)} + \dots + D^{(N)}) \circ V_N + \frac{N(N-1)}{2}, \quad (3.4)$$

correctly determines a linear operator $\text{Sym}(N) \rightarrow \text{Sym}(N)$, and we have

$$D_N \phi_{\nu|N} = -|\nu| \phi_{\nu|N}, \quad \forall \nu \in \mathbb{Y}(N). \quad (3.5)$$

In particular, D_N annihilates the constants.

Note that in the simplest case $N = 1$, operator D_N reduces to D .

Proof. Multiplication by V_N takes a symmetric polynomial to an antisymmetric one. Application of the symmetric operator $\sum D_{x_i}$ produces another antisymmetric polynomial. It can be divided by V_N , which finally results in another symmetric polynomial. Therefore, D_N is correctly defined in $\text{Sym}(N)$.

It follows from (3.1) that the operator $V_N^{-1} \circ (\sum D_{x_i}) \circ V_N$ multiplies ϕ_{ν} by

$$-\sum_{i=1}^N (\nu_i + N - i) = -|\nu| - \frac{N(N-1)}{2},$$

and then the second term is cancelled by the constant term $N(N-1)/2$ in D_N , leading finally to $-|\nu|$, as stated.

In particular, $D_N \phi_{\emptyset} = 0$, which means that $D_N 1 = 0$. \square

For $D = D^L$ or $D = D^M$, we will denote the corresponding operators $D_N : \text{Sym}(N) \rightarrow \text{Sym}(N)$ by D_N^L or D_N^M , respectively. The relation (3.5) now turns into

$$D_N^L L_{\nu|N,b} = -|\nu| L_{\nu|N,b}, \quad D_N^M M_{\nu|N,b,\xi} = -|\nu| M_{\nu|N,b,\xi}, \quad \nu \in \mathbb{Y}(N). \quad (3.6)$$

In the next proposition we interpret D_N^M as a partial difference operator living on the discrete set formed by ordered N -tuples of points in \mathbb{Z}_+^N ,

$$\mathbb{Z}_{+, \text{ord}}^N := \{x = (x_1, \dots, x_N) \in \mathbb{Z}_+^N : x_1 > \dots > x_N\}. \quad (3.7)$$

Note that any polynomial in variables x_1, \dots, x_N is uniquely determined by its restriction to $\mathbb{Z}_{+, \text{ord}}^N$.

Proposition 3.4. *Upon restriction to $\mathbb{Z}_{+, \text{ord}}^N$, the operator $D_N^M : \text{Sym}(N) \rightarrow \text{Sym}(N)$ is implemented by the difference operator*

$$\begin{aligned} D_N^M f(x) &= \sum_{i=1}^N A_i(x) f(x + \varepsilon_i) + \sum_{i=1}^N B_i(x) f(x - \varepsilon_i) - C(x) f(x) \\ &= \sum_{i=1}^N A_i(x) (f(x + \varepsilon_i) - f(x)) + \sum_{i=1}^N B_i(x) (f(x - \varepsilon_i) - f(x)). \end{aligned} \quad (3.8)$$

Here $f(x)$ is an arbitrary function on $\mathbb{Z}_{+, \text{ord}}^N$, $\{\varepsilon_1, \dots, \varepsilon_N\}$ is the canonical basis in \mathbb{R}^N , and the coefficients are given by

$$\begin{aligned} A_i(x) &= \frac{\xi(b+x_i)}{1-\xi} \frac{V_N(x+\varepsilon_i)}{V_N(x)}, \\ B_i(x) &= \frac{x_i}{1-\xi} \frac{V_N(x-\varepsilon_i)}{V_N(x)} \\ C(x) &= \frac{\xi b N + (1+\xi) \sum_{i=1}^N x_i}{1-\xi} - \frac{N(N-1)}{2}, \end{aligned} \tag{3.9}$$

where $V_N(x) = V_N(x_1, \dots, x_N)$.

Note that if $x + \varepsilon_i$ or $x - \varepsilon_i$ falls outside $\mathbb{Z}_{+, \text{ord}}^N(N)$, then the corresponding coefficient automatically vanishes.

Proof. Easy direct check. \square

Likewise, the next proposition says that D_N^{L} can be explicitly written as a partial differential operator.

Proposition 3.5. *Upon restriction to the open cone*

$$\mathbb{R}_{>0, \text{ord}}^N := \{(x_1, \dots, x_N) \in \mathbb{R}^N : x_1 > \dots > x_N > 0\}$$

the operator $D_N^{\text{M}} : \text{Sym}(N) \rightarrow \text{Sym}(N)$ is implemented by the partial differential operator

$$D_N^{\text{L}} = \sum_{i=1}^N x_i \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^N \left(b - x_i + \sum_{j:j \neq i} \frac{2x_i}{x_i - x_j} \right) \frac{\partial}{\partial x_i},$$

Proof. Direct computation. \square

Finally, write down the action of D_N^{L} and D_N^{M} on the Schur polynomials and on the factorial Schur polynomials, respectively:

For $\nu \in \mathbb{Y}(N)$ and $i = 1, \dots, N$, denote by $\nu - \varepsilon_i$ the vector obtained from $\nu = (\nu_1, \dots, \nu_N)$ by decreasing the i th coordinate by 1. If $\nu - \varepsilon_i$ is not a partition, then we agree that $S_{\nu - \varepsilon_i|N} = S_{\nu - \varepsilon_i|N}^{\text{f}} = 0$. As above, we set $n_i = \nu_i + N - i$.

Proposition 3.6. *In this notation, we have*

$$D_N^{\text{L}} S_{\nu|N} = -|\nu| S_{\nu|N} + \sum_{i=1}^N n_i (n_i + b - 1) S_{\nu - \varepsilon_i|N} \tag{3.10}$$

$$D_N^{\text{M}} S_{\nu|N}^{\text{f}} = -|\nu| S_{\nu|N}^{\text{f}} + \frac{\xi}{1-\xi} \sum_{i=1}^N n_i (n_i + b - 1) S_{\nu - \varepsilon_i|N}^{\text{f}}. \tag{3.11}$$

Proof. Let us check the first relation. By (3.4), $D_N^L S_{\nu|N}$ is the sum of two terms: one is $\frac{N(N-1)}{2} S_{\nu|N}$ and the other is

$$\{V_N^{-1} \circ (D^{(1)} + \dots + D^{(N)}) \circ V_N\} S_{\nu|N},$$

where $D^{(1)}, \dots, D^{(N)}$ are copies of the one-dimensional Laguerre operator D^L acting on variables x_1, \dots, x_N , respectively. By the definition of $S_{\nu|N}$, the last expression equals

$$\frac{(D^{(1)} + \dots + D^{(N)}) \det [x_j^{n_k}]_{j,k=1}^N}{V_N(x)}.$$

Expanding the determinant and applying formula (2.2) for the action of D^L on monomials in one variable, we get after simple transformations

$$-(n_1 + \dots + n_N) \frac{\det [x_j^{n_k}]_{j,k=1}^N}{V_N(x)} + \sum_{i=1}^N n_i (n_i + b - 1) \frac{\det [x_j^{n_k - \delta_{ik}}]_{j,k=1}^N}{V_N(x)}.$$

Since

$$-(n_1 + \dots + n_N) + \frac{N(N-1)}{2} = -|\nu|,$$

this gives the desired result.

The second relation is checked in the same way. \square

3.4. Orthogonality. Let again $\{\phi_n\}$ be a sequence of real monic polynomials with $\deg \phi_n = n$. Assume that there exists a probability measure w on \mathbb{R} with finite moments of all orders and such that the polynomials ϕ_n form an orthogonal system in the weighted Hilbert space $L^2(\mathbb{R}, w)$:

$$(\phi_m, \phi_n) := \int \phi_m(x) \phi_n(x) w(dx) = \delta_{mn} \cdot \text{const}_n, \quad m, n \in \mathbb{Z}_+.$$

Given an arbitrary $N = 1, 2, \dots$, we use the shorthand notation $x := (x_1, \dots, x_N)$, $V_N(x) := V_N(x_1, \dots, x_N)$, and set

$$\mathbb{R}_{\text{ord}}^N := \{x = (x_1, \dots, x_N) \in \mathbb{R}^N : x_1 > \dots > x_N\}.$$

Points of $\mathbb{R}_{\text{ord}}^N$ can be interpreted as N -particle configurations in \mathbb{R} .

Next, introduce an inner product in $\text{Sym}(N)$ by setting for $F, G \in \text{Sym}(N)$

$$(F, G)_N = \frac{1}{N!} \prod_{i=1}^N \frac{1}{(\phi_{N-i}, \phi_{N-i})} \int_{\mathbb{R}^N} F(x) G(x) V_N^2(x) \prod w(dx_i) \quad (3.12)$$

$$= \prod_{i=1}^N \frac{1}{(\phi_{N-i}, \phi_{N-i})} \int_{R_{\text{ord}}^N} F(x) G(x) V_N^2(x) \prod w(dx_i) \quad (3.13)$$

The expressions in (3.12) and (3.13) are the same for the following reasons: First, since $V_N(x)$ vanishes on the diagonal hyperplanes $x_i = x_j$, we may restrict integration in (3.12) to the complement to all such hyperplanes, even if the measure w has atoms. Second, since F and G are symmetric, we may further restrict integration to $\mathbb{R}_{\text{ord}}^N$ by introducing the extra factor $N!$, which will cancel the same factorial in the denominator.

Proposition 3.7. *The N -variate symmetric polynomials $\phi_{\nu|N}$ are orthogonal with respect to the above inner product. More precisely,*

$$(\phi_{\mu|N}, \phi_{\nu|N})_N = \delta_{\mu\nu} \cdot \prod_{i=1}^N \frac{(\phi_{n_i}, \phi_{n_i})}{(\phi_{N-i}, \phi_{N-i})} \quad (3.14)$$

where $\mu, \nu \in \mathbb{Y}(N)$ and $n_i = \nu_i + N - i$.

Note that for the constant function $\phi_{\emptyset|N} \equiv 1$, one has $n_i = N - i$, so that the squared norm equals $(\phi_{\emptyset|N}, \phi_{\emptyset|N})_N = 1$.

Proof. Direct computation: Use the definition of the polynomials as ratios of determinants and observe that the V_N^2 factor in the integrand cancels the denominators. Then expand the determinants in the numerators and integrate out taking into account orthogonality of the univariate polynomials. \square

Obviously, the inner product $(F, G)_N$ in $\text{Sym}(N)$ defined above coincides with the integral of FG against the following measure on $\mathbb{R}_{\text{ord}}^N$

$$w_N := \prod_{i=1}^N \frac{1}{(\phi_{N-i}, \phi_{N-i})} \cdot V_N^2 \prod w(dx_i) \Big|_{\mathbb{R}_{\text{ord}}^N}. \quad (3.15)$$

Since $(1, 1)_N = 1$, w_N is a probability measure on $\mathbb{R}_{\text{ord}}^N$. The probability space $(\mathbb{R}_{\text{ord}}^N, w_N)$ gives rise to random N -particle configurations that constitute the *N -particle orthogonal polynomial ensemble* corresponding to the system $\{\phi_n\}$ of orthogonal polynomials with weight w , see [Ko05].

Proposition 3.8. *Assume w is such that the space of polynomials is dense in $L^2(\mathbb{R}, w)$. Then the N -variate polynomials ϕ_{ν} , $\nu \in \mathbb{Y}(N)$, form an orthogonal basis in $L^2(\mathbb{R}_{\text{ord}}^N, w_N)$ for every N .*

Proof. The operator of multiplication by V_N maps $L^2(\mathbb{R}_{\text{ord}}^N, w_N)$ isometrically onto the Hilbert space

$$L^2(\mathbb{R}_{\text{ord}}^N, \text{const} \prod w(dx_i))$$

with an appropriate constant factor in front of the product measure $\prod w(dx_i)$. Further, the latter space can be identified with the subspace of antisymmetric functions in the Hilbert space $L^2(\mathbb{R}^N, \text{const} \prod w(dx_i))$, where antisymmetry is understood with respect to the action of the symmetric group S_n by permutations of coordinates. By the assumption on w , any function in the latter Hilbert space can be approximated,

in the Hilbert norm, by N -variate polynomials. Applying antisymmetrization, we get that any antisymmetric function can be approximated by antisymmetric polynomials. Finally, any such polynomial is the product of a symmetric polynomial by V_N . This concludes the proof. \square

Now we specialize the proposition to the case of Laguerre and Meixner polynomials; then we take as w the gamma distribution on $\mathbb{R}_+ \subset \mathbb{R}$ or the negative binomial distribution on $\mathbb{Z}_+ \subset \mathbb{R}$, respectively. Note that in the latter case w is purely atomic, so that the remark after (3.13) concerning the possibility to remove the hyperplanes $x_i = x_j$ becomes really meaningful. We also use the explicit formulas (2.4) and (2.7) for the squared norms. This leads to the following claim:

Corollary 3.9. *For the N -variate Laguerre and Meixner symmetric polynomials we have*

$$(L_{\mu|N,b}, L_{\nu|N,b})_N = \delta_{\mu\nu} \cdot \prod_{i=1}^N (N - i + 1)_{\nu_i} (N + b - i)_{\nu_i} \quad (3.16)$$

$$(M_{\mu|N,b,\xi}, M_{\nu|N,b,\xi})_N = \delta_{\mu\nu} \cdot \frac{\xi^{|\nu|}}{(1 - \xi)^{2|\nu|}} \prod_{i=1}^N (N - i + 1)_{\nu_i} (N + b - i)_{\nu_i} \quad (3.17)$$

4. THE LAGUERRE AND MEIXNER SYMMETRIC FUNCTIONS

4.1. Preliminaries on the algebra Sym of symmetric functions. So far we were dealing with partitions ν of restricted length, but in what follows we consider arbitrary partitions and we agree to identify them with Young diagrams. The set of all partitions (=Young diagrams) will be denoted by \mathbb{Y} . We denote by $\ell(\nu)$ the length of ν , that is, the number of nonzero coordinates (or nonzero rows).

Definition 4.1. For every $N \geq 1$, consider the algebra morphism

$$\pi_{N-1,N}: \text{Sym}(N) \rightarrow \text{Sym}(N-1)$$

defined by

$$(\pi_{N-1,N} f)(x_1, \dots, x_{N-1}) = f(x_1, \dots, x_{N-1}, 0).$$

Let Sym consist of all sequences $(f_N \in \text{Sym}(N))$ such that, first, $\pi_{N-1,N} f_N = f_{N-1}$ for each N and, second, $\sup \deg f_N < \infty$. This is an algebra under termwise operations; it is called the *algebra of symmetric functions*. An element $f = (f_N)$ is said to be homogeneous of degree k if so are all f_N 's. The canonical morphism $\text{Sym} \rightarrow \text{Sym}(N)$ taking f to f_N will be denoted by π_N .

The above definition follows [Ma95] (only we are working over \mathbb{R} , not \mathbb{Z}). Equivalently, one can say that Sym is the subalgebra in $\mathbb{R}[[x_1, x_2, \dots]]$ formed by symmetric formal power series of bounded degree, see [Sa01].

By the very definition, Sym is a graded algebra. There are three distinguished systems of algebraically independent homogeneous generators of Sym : the *Newton power sums*

$$p_k = \sum_{i=1}^{\infty} x_i^k, \quad k = 1, 2, \dots,$$

the *elementary symmetric functions*

$$e_k = \sum_{1 \leq i_1 < \dots < i_k} x_{i_1} \dots x_{i_k}, \quad k = 1, 2, \dots,$$

and the *complete homogeneous symmetric functions*

$$h_k = \sum_{1 \leq i_1 \leq \dots \leq i_k} x_{i_1} \dots x_{i_k}, \quad k = 1, 2, \dots$$

Thus, Sym may be identified with the algebra of polynomials in each of these three systems of variables,

$$\text{Sym} = \mathbb{R}[p_1, p_2, \dots] = \mathbb{R}[e_1, e_2, \dots] = \mathbb{R}[h_1, h_2, \dots],$$

with the understanding that $\deg p_k = \deg e_k = \deg h_k = k$.

The Schur polynomials are stable in the sense that

$$S_{\nu|N}|_{x_N=0} = S_{\nu|N-1}, \quad (4.1)$$

where by convention $S_{\nu|N} \equiv 0$ if $N < \ell(\nu)$. This make it possible to define their analogs in the algebra Sym , the *Schur symmetric functions*

$$S_{\nu} = \varprojlim S_{\nu|N}, \quad \nu \in \mathbb{Y},$$

characterized by

$$\pi_N(S_{\nu}) = S_{\nu|N} \quad \forall N.$$

The Schur functions S_{ν} form a homogeneous basis in Sym . They are expressed through $\{e_k\}$ and $\{h_k\}$ as follows

$$S_{\nu} = \det[e_{\nu'_i - i + j}] = \det[h_{\nu_i - i + j}],$$

where ν' is the transposed Young diagram and $e_0 = h_0 = 1$, $e_{-k} = h_{-k} = 0$ for $k = 1, 2, \dots$; the order of determinants may be taken arbitrarily provided it is large enough.

In the initial definition, the algebra Sym is tied to an infinite collection of variables x_1, x_2, \dots , but often it is preferable to adopt a different point of view and interpret symmetric functions as polynomials in $\{p_k\}$ or $\{h_k\}$ or $\{e_k\}$, which creates extra degrees of freedom and leads to useful realizations of the algebra Sym , other than its initial realization inside $\mathbb{R}[[x_1, x_2, \dots]]$.

An important example, which we substantially exploit below, is the *supersymmetric realization* of Sym inside the algebra $\mathbb{R}[[x_1, x_2, \dots; y_1, y_2, \dots]]$ of formal series in a doubly infinite collection of variables. In this realization,

$$p_k \rightarrow \sum_{i=1}^{\infty} x_i^k + (-1)^{k-1} \sum_{i=1}^{\infty} y_i^k, \quad k = 1, 2, \dots$$

Let $\sigma : \text{Sym} \rightarrow \text{Sym}$ denote the involutive linear map given by $\sigma(S_\nu) = S_{\nu'}$ for each $\nu \in \mathbb{Y}$. (In [Ma95], it is denoted by ω .) This is an algebra automorphism interchanging h_k with e_k and taking p_k to $(-1)^{k-1} p_k$. In the supersymmetric realization, the involution σ is implemented by the automorphism of $\mathbb{R}[[x_1, x_2, \dots; y_1, y_2, \dots]]$ interchanging x_i with y_i for all i , whereas in the standard realization $\text{Sym} \subset \mathbb{R}[x_1, x_2, \dots]$, such a natural interpretation is lacking.

4.2. The Laguerre symmetric functions. Return to the expansion (3.2) and examine the expression

$$\prod_{i=1}^N \frac{(\nu_i + N - i)!(b)_{\nu_i + N - i}}{(\mu_i + N - i)!(b)_{\mu_i + N - i}} = \prod_{i=1}^N \frac{n_i!(b)_{n_i}}{m_i!(b)_{m_i}}$$

for the coefficients in front of the Schur polynomials. Recall that $\nu \in \mathbb{Y}(N)$ and $\mu \subseteq \nu$; as before, we also use the shorthand notation $n_i = \nu_i + N - i$, $m_i = \mu_i + N - i$.

We are going to rewrite this expression as a product over the boxes of the skew Young diagram ν/μ . We need an extra notation: given a box $\square = (i, j)$ (meaning that i and j are the row and column numbers of \square), the difference $j - i$ is called the *content* of \square and is denoted as $c(\square)$. The following is a generalization of the Pochhammer symbol:

$$(z)_{\nu/\mu} = \prod_{\square \in \nu/\mu} (z + c(\square)), \quad z \in \mathbb{C}.$$

In the particular case when $\mu = \emptyset$ and $\nu = (n)$ or $\nu = (1^n)$, this gives $(z)_n$ or $z^{\downarrow n}$, respectively.

Lemma 4.2. *With the above notation,*

$$\prod_{i=1}^N \frac{n_i!(b)_{n_i}}{m_i!(b)_{m_i}} = (N)_{\nu/\mu} (N + b - 1)_{\nu/\mu}. \quad (4.2)$$

Proof. The i th term entering the product in left-hand side equals

$$(m_i + 1) \dots n_i \cdot (m_i + b) \dots (n_i + b - 1),$$

which is the same as the product in the right-hand side restricted to the boxes entering the i th row of ν/μ . \square

Despite the different origin of N and b , the quantities N and $N + b - 1$ enter the right-hand side of (4.2) symmetrically. This observation is the starting point for the next definition.

Definition 4.3. Let z and z' be complex parameters and $\text{Sym}_{\mathbb{C}} = \text{Sym} \otimes_{\mathbb{R}} \mathbb{C}$ denote the complexification of the algebra Sym . The *Laguerre symmetric function* $\mathfrak{L}_{\nu} \in \text{Sym}_{\mathbb{C}}$ with index $\nu \in \mathbb{Y}$ and parameters (z, z') is defined by the following expansion in the basis of the Schur symmetric functions:

$$\mathfrak{L}_{\nu} = \sum_{\mu: \mu \subseteq \nu} (-1)^{|\nu| - |\mu|} \frac{\dim \nu / \mu}{(|\nu| - |\mu|)!} (z)_{\nu/\mu} (z')_{\nu/\mu} S_{\mu}. \quad (4.3)$$

That is, we formally rename N and $N + b - 1$ by z and z' , and replace in (3.2) the Schur polynomials by the Schur symmetric functions.

Clearly, \mathfrak{L}_{ν} is an inhomogeneous element of Sym of degree $|\nu|$, with top degree homogeneous component equal to S_{ν} . Let $\text{Sym}^{\deg \leq n} \subset \text{Sym}$ denote the subspace of elements of degree less or equal to n . Clearly, for each natural n , the elements \mathfrak{L}_{ν} with $|\nu| \leq n$ form a basis in $\text{Sym}^{\deg \leq n}$. Thus, $\{\mathfrak{L}_{\nu}\}$ is a basis in Sym consistent with the canonical filtration of Sym determined by its graduation.

Note that, since parameters z and z' enter (4.3) in a symmetric way, \mathfrak{L}_{ν} is not affected by the transposition $z \leftrightarrow z'$.

One more remark is that

$$\sigma(\mathfrak{L}_{\nu}) = \mathfrak{L}_{\nu'} \Big|_{z \rightarrow -z, z' \rightarrow -z'}.$$

In words: application of the involution σ to \mathfrak{L}_{ν} results in transposition of the index ν and multiplication of the parameters z and z' by -1 . This nice symmetry relation appears only on the level of the algebra Sym ; in the finite-variate case it does not exist.

4.3. Analytic continuation. Let us start with a few claims which are obvious consequences of the results of the previous two subsections.

Let $J_N \subset \text{Sym}$ denote the kernel of the projection $\pi_N : \text{Sym} \rightarrow \text{Sym}(N)$; J_N is the ideal in Sym generated by elements e_k with indices $k > N$.

Proposition 4.4. *For every $N = 1, 2, \dots$, the Schur functions S_{ν} with $\ell(\nu) > N$ form a basis in J_N , while the remaining Schur functions span a complement to J_N .*

Recall that $\text{Sym}^{\deg \leq n} \subset \text{Sym}$ denotes the finite-dimensional subspace formed by elements of degree less or equal to n .

Corollary 4.5. *Fix an arbitrary $n = 1, 2, \dots$. For N large enough, one has $J_N \cap \text{Sym}^{\deg \leq n} = \{0\}$.*

More precisely, the above relation holds starting from $N = n$.

The following proposition provides a characterization of the Laguerre symmetric functions and explains the origin of their definition:

Proposition 4.6. *For any fixed $\nu \in \mathbb{Y}$, the Laguerre symmetric function \mathfrak{L}_ν can be characterized as the only element of the algebra*

$$\text{Sym}[z, z'] = \text{Sym} \otimes \mathbb{C}[z, z']$$

such that for any natural N and any $b > 0$ one has

$$\pi_N \left(\mathfrak{L}_\nu \big|_{z=N, z'=N+b-1} \right) = \begin{cases} L_{\nu|N, b}, & \text{if } \ell(\nu) \leq N \\ 0, & \text{if } \ell(\nu) > N. \end{cases}$$

Proof. Recall that

$$\pi_N \left(S_\nu \big|_{z=N, z'=N+b-1} \right) = \begin{cases} S_{\nu|N}, & \text{if } \ell(\nu) \leq N \\ 0, & \text{if } \ell(\nu) > N. \end{cases}$$

Using this and comparing (4.3) and (3.2) we see that \mathfrak{L}_ν has the required property.

Let us prove the uniqueness statement. It says that if an element $F \in \text{Sym}[z, z']$ is such that

$$\pi_N \left(F \big|_{z=N, z'=N+b-1} \right) = 0$$

for all N and b as above, then $F = 0$.

Fix n so large that

$$F \in \text{Sym}^{\deg \leq n}[z, z'].$$

By virtue of Corollary 4.5, for large N , a stronger condition holds:

$$F \big|_{z=N, z'=N+b-1} = 0$$

This implies $F \equiv 0$, because F is a (vector-valued) polynomial in (z, z') , and any point set of the form

$$\{(z, z') = (N, N + b - 1) : N = N_0, N_0 + 1, N_0 + 2, \dots, \quad b > 0\} \subset \mathbb{C}^2,$$

is a uniqueness set for polynomials in two variables. \square

Informally, Proposition 4.6 may be interpreted as follows:

The Laguerre symmetric functions (4.3) are obtained from the N -variate Laguerre symmetric polynomials (3.2) by analytic continuation with respect to parameters N and b .

As we will see, Proposition 4.6 makes it possible to prove various algebraic relations involving the Laguerre symmetric functions using the principle of analytic continuation of identities. The crucial fact is that, by virtue of Lemma 4.2, the dependence of (3.2) in N and b is polynomial, which makes extrapolation from discrete values of parameter N to the complex domain unambiguous.

4.4. The Laguerre differential operator \mathfrak{D}^L : $\text{Sym}_{\mathbb{C}} \rightarrow \text{Sym}_{\mathbb{C}}$. A box \square in a Young diagram ν is said to be a *corner box* if the shape $\nu \setminus \square$ obtained by removing \square from ν is again a Young diagram. By ν^- we denote the set of all corner boxes in ν . For instance, if $\nu = (3, 2, 2)$ then ν^- comprises two corner boxes, $\square = (1, 3)$ and $\square = (3, 2)$.

Definition 4.7. Introduce the *Laguerre operator* \mathfrak{D}^L : $\text{Sym}_{\mathbb{C}} \rightarrow \text{Sym}_{\mathbb{C}}$ with parameters z, z' by setting

$$\mathfrak{D}^L S_{\nu} = -|\nu| S_{\nu} + \sum_{\square \in \nu^-} (z + c(\square))(z' + c(\square)) S_{\nu \setminus \square} \quad (4.4)$$

This formula is obtained from (3.10) by formal substitution

$$N \rightarrow z, \quad N + b - 1 \rightarrow z'.$$

Note that the resulting expression for \mathfrak{D}^L is symmetric under $z \leftrightarrow z'$. Note also that

$$\sigma \circ \mathfrak{D}^L \circ \sigma = \mathfrak{D}^L \Big|_{z \rightarrow -z, z' \rightarrow -z'}. \quad (4.5)$$

Proposition 4.8. *If $z = N = 1, 2, \dots$ and $z' = N + b - 1$ with $b > 0$, then the operator \mathfrak{D}^L preserves the ideal $J_N \subset \text{Sym}$, and its action on the quotient space $\text{Sym}/J_N = \text{Sym}(N)$ coincides with that of the N -variate Laguerre operator D_N^L with parameter b .*

Proof. Let us prove the first claim: $\mathfrak{D}^L J_N \subseteq J_N$. Recall that J_N is spanned by the Schur functions S_{ν} with $\ell(\nu) \geq N + 1$. Thus, it suffices to check that $\mathfrak{D}^L S_{\nu} \in J_N$ provided that $\ell(\nu) \geq N + 1$. By the very definition, $\mathfrak{D}^L S_{\nu}$ is a linear combination of S_{ν} and the elements of the form $S_{\nu \setminus \square}$. We have $\ell(\nu \setminus \square) \geq N + 1$ with the only exception when $\ell(\nu) = N + 1$, $\nu_{N+1} = 1$, and $\square = (N + 1, 1)$. But in this case the factor $z + c(\square) = N + c(\square)$ in front of $S_{\nu \setminus \square}$ vanishes, because $c(\square) = -N$.

The second claim is obvious, because the action of the operator \mathfrak{D}^L with parameters $z = N$ and $z' = N + b - 1$ on the Schur functions S_{ν} with $\ell(\nu) \leq N$ is exactly the same as the action of the operator D_N on the N -variate Schur polynomials $S_{\nu|N} = \pi_N(S_{\nu})$. \square

Proposition 4.9. *The Laguerre symmetric functions are eigenvectors of the operator \mathfrak{D}^L : we have $\mathfrak{D}^L \mathfrak{L}_{\nu} = -|\nu| \mathfrak{L}_{\nu}$ for every $\nu \in \mathbb{Y}$.*

Proof. We argue as in the proof of Proposition 4.6. Both sides of the equality in question are elements of a space of vector-valued polynomials,

$$\text{Sym}^{\deg \leq |\nu|}[z, z'] = \text{Sym}^{\deg \leq |\nu|} \otimes \mathbb{C}[z, z'].$$

When $z = N$ and $z' = N + b - 1$, the desired equality holds after factorization modulo J_N : this follows from Proposition 4.8, because the similar equality holds in the N -variable case, see (3.10). Next, the same argument as above shows that the equality actually holds without factorization, provided that N is large enough. This suffices to conclude that the equality holds for all complex values of z and z' . \square

Further, observe that any linear operator in an algebra of polynomials (with finitely or countably many variables) can be represented as a formal differential operator with polynomial coefficients. In the next theorem we describe such a presentation for \mathfrak{D}^L .

Theorem 4.10. *Upon the identification $\text{Sym}_{\mathbb{C}} = \mathbb{C}[e_1, e_2, \dots]$, the Laguerre operator $\mathfrak{D}^L : \text{Sym}_{\mathbb{C}} \rightarrow \text{Sym}_{\mathbb{C}}$ with parameters z and z' can be written as a second order differential operator in variables e_1, e_2, \dots ,*

$$\begin{aligned} \mathfrak{D}^L = & \sum_{n \geq 1} \left(\sum_{k=0}^{n-1} (2n-1-2k) e_{2n-1-k} e_k \right) \frac{\partial^2}{\partial e_n^2} \\ & + 2 \sum_{m > n \geq 1} \left(\sum_{k=0}^{n-1} (m+n-1-2k) e_{m+n-1-k} e_k \right) \frac{\partial^2}{\partial e_m \partial e_n} \\ & + \sum_{n=1}^{\infty} (-n e_n + (z-n+1)(z'-n+1) e_{n-1}) \frac{\partial}{\partial e_n} \end{aligned}$$

with the understanding that $e_0 = 1$.

First, we will prove two lemmas.

Lemma 4.11. *Temporarily denote by X the formal differential operator in the right-hand side and regard X as an operator acting in $\text{Sym}_{\mathbb{C}}$. If $z = N = 1, 2, \dots$, then X preserves the ideal $J_N \subset \text{Sym}$.*

Proof. Split X into the sum $X_2 + X_1$ of the second and first order terms. We use the fact that J_N is generated by the elements e_n with $n \geq N+1$. This means that J_N is spanned by those monomials in the generators e_1, e_2, \dots that contain at least one letter e_n with $n \geq N+1$. Take any such monomial and apply to it the operator X_2 . From the explicit form of X_2 it is seen that if a letter e_n disappears after application of a differential monomial entering the formula, then another letter $e_{n'}$ with index $n' \geq n$ comes from the coefficient of this monomial, so that the result is again contained in J_N .

Obviously, the same happens after application of the term $\sum (-e_n) \partial / \partial e_n$ in X_1 .

Examine now the remaining first order term in X_1 . Its application to e_n lowers the index n by 1, so a possible problem arises when $n = N+1$. But then the coefficient $(z-n+1)(z'-n+1)$ vanishes because, by the assumption, z is specialized to N . \square

Keep the notation X for the above differential operator, fix $N = 1, 2, \dots$, and denote by X_N the differential operator in variables e_1, \dots, e_N that is obtained from operator X by the following truncation procedure: first, keep only terms which do not contain derivatives on variables e_n with $n \geq N+1$, next, put $e_{N+1} = e_{N+2} = \dots = 0$ in the coefficients, and, finally, specialize $z = N$ and $z' = N + b - 1$.

Lemma 4.12. *Identify $\text{Sym}(N)$ with the algebra $\mathbb{R}[e_1, \dots, e_N]$. Then the operator $D_N^L : \text{Sym}(N) \rightarrow \text{Sym}(N)$ with parameter $b > 0$ coincides with the operator X_N just defined.*

Proof. We start by recalling a well-known abstract formalism. Let \mathcal{A} be a commutative unital algebra and $\mathcal{X} : \mathcal{A} \rightarrow \mathcal{A}$ be a linear operator. For $a \in \mathcal{A}$, denote by M_a the operator of multiplication by a . Let us say that \mathcal{X} has order $\leq k$ if its $(k+1)$ -fold commutator with operators of multiplication by arbitrary elements of the algebra vanishes:

$$[M_{a_1}, [M_{a_2}, \dots [M_{a_{k+1}}, \mathcal{X}] \dots]] = 0, \quad a_1, \dots, a_{k+1} \in \mathcal{A}.$$

If X has order ≤ 0 then $\mathcal{X} = M_a$, where $a = X1$. If \mathcal{X} has order $\leq k$ with $k \geq 1$ and $a \in \mathcal{A}$ is arbitrary, then $[M_a, \mathcal{X}]$ has order $\leq k-1$. Using this, it is an easy exercise to check that if \mathcal{A} is generated by a sequence of elements a_1, a_2, \dots , then any operator \mathcal{X} of order $\leq k$ is uniquely determined by its action on monomials of degree $\leq k$ in the generators.

Obviously, a differential operator of order k has order $\leq k$ in this abstract sense. Consequently, the operator D_N^L has order ≤ 2 in the abstract sense, since it can be written as a second order differential operator (the fact that the coefficients have singularities along the hyperplanes $x_i = x_j$ is inessential here).

Return now to the equality in question, $D_N^L = X_N$. Both D_N^L and X_N are operators in $\mathbb{R}[e_1, \dots, e_N]$ of order ≤ 2 in the abstract sense. Therefore, it suffices to verify that they coincide on 1, on the generators e_n , and on quadratic elements $e_m e_n$, where $m, n = 1, \dots, N$.

The first assertion is obvious, as both operators annihilate 1.

The second assertion means that

$$D_N^L e_n = -e_n + (N - n + 1)(N - n + b)e_{n-1}.$$

This is clear from (3.10), because $e_n = S_{(1^n)}$.

The third assertion amounts to the equality

$$\frac{1}{2} (D_N^L(e_m e_n) - (D_N^L e_m)e_n - e_m(D_N^L e_n)) = \sum_{k,l} (k-l)e_k e_l,$$

summed over couples $k > l$ such that $k+l = m+n-1$, $0 \leq l \leq n-1$, and $k \leq N$. Here we apply the formula

$$e_m e_n = \sum S_\nu,$$

where summation is over two-column diagrams ν such that $\nu' = (m+r, n-r)$, where $0 \leq r \leq n$ and $m+r \leq N$. This allows us to apply formula (3.10). Then we use the fact that if ν is a two-column diagram and $\nu' = (p, q)$, then

$$S_\nu = e_p e_q - e_{p+1} e_{q-1}$$

with the understanding that $e_0 = 1$ and $e_{-1} = e_{N+1} = e_{N+2} = \dots = 0$. Then there are many cancellations and finally we get the desired result. \square

Proof of the theorem. Let X be the same differential operator as above. We will prove that $\mathfrak{D}^L F = XF$ for every $F \in \text{Sym}$ by using the principle of analytic continuation.

As in the proof of Proposition 4.6, it is enough to establish the equality

$$\pi_N \left(\mathfrak{D}^L \Big|_{z=N, z'=N+b-1} F \right) = \pi_N \left(X \Big|_{z=N, z'=N+b-1} F \right).$$

Set $F_N = \pi_N(F)$. By Proposition 4.8, the left-hand side equals $D_N^L F_N$. By Lemma 4.11, the right-hand side equals $X_N F_N$. Finally, the Lemma 4.12 says that $D_N^L = X_N$, which concludes the proof. \square

Corollary 4.13. *Upon the identification $\text{Sym}_{\mathbb{C}} = \mathbb{C}[h_1, h_2, \dots]$, the Laguerre operator $D^L : \text{Sym}_{\mathbb{C}} \rightarrow \text{Sym}_{\mathbb{C}}$ with parameters z and z' can be represented as a second order differential operator in variables h_1, h_2, \dots ,*

$$\begin{aligned} \mathfrak{D}^L = & \sum_{n \geq 1} \left(\sum_{k=0}^{n-1} (2n-1-2k) h_{2n-1-k} h_k \right) \frac{\partial^2}{\partial h_n^2} \\ & + 2 \sum_{m > n \geq 1} \left(\sum_{k=0}^{n-1} (m+n-1-2k) h_{m+n-1-k} h_k \right) \frac{\partial^2}{\partial h_m \partial h_n} \\ & + \sum_{n=1}^{\infty} \left(-h_n + (z+n-1)(z'+n-1) h_{n-1} \right) \frac{\partial}{\partial h_n} \end{aligned}$$

with the understanding that $h_0 = 1$.

Proof. Since the involution $\sigma : \text{Sym} \rightarrow \text{Sym}$ acts as $e_n \leftrightarrow h_n$, this follows from (4.5). \square

Remark 4.14. In terms of the generators p_1, p_2, \dots , the Laguerre operator is given by

$$\begin{aligned} \mathfrak{D}^L = & \sum_{i=1}^{\infty} \left(-ip_i \frac{\partial}{\partial p_i} + (z+z')(i+1)p_i \frac{\partial}{\partial p_{i+1}} \right) + zz' \frac{\partial}{\partial p_1} \\ & + \sum_{i,j=1}^{\infty} \left(ij p_{i+j-1} \frac{\partial}{\partial p_i} \frac{\partial}{\partial p_j} + (i+j+1)p_i p_j \frac{\partial}{\partial p_{i+j+1}} \right). \end{aligned} \tag{4.6}$$

This formula is obtained by combining (4.4) with [BO09, Lemma 6.3]; it agrees with the formula given in [DH11, Definition 3.5].

Note that the Laguerre symmetric functions can be characterized by the following two properties (cf. [La91c]): first, \mathfrak{L}_{ν} differs from S_{ν} by lower degree terms; second, \mathfrak{L}_{ν} is an eigenvector of the operator \mathfrak{D}^L (necessarily, with the eigenvalue $-|\nu|$).

4.5. Shifted symmetric functions. We aim at applying the same principle of analytic continuation to constructing symmetric functions analogs of the Meixner polynomials. Recall that our starting point was the expansion formula (3.2). Its counterpart is (3.3), so we need symmetric functions analogs of the factorial Schur polynomials $S_{\mu|N}^f$ entering (3.3). However, we cannot directly imitate the above definition of the Schur functions via the Schur polynomials because the factorial Schur polynomials do not share the stability property (4.1) of the ordinary Schur polynomials: Indeed, instead of (4.1), the following relation holds

$$S_{\nu|N}^f(x_1, \dots, x_N)|_{x_N=0} = S_{\nu|N-1}^f(x_1 - 1, \dots, x_{N-1} - 1) \quad (4.7)$$

(as above, we agree that $S_{\nu|N}^f \equiv 0$ when $\ell(\nu) > N$).

This circumstance forces us to take a roundabout way. Following [OO97a], say that a polynomial $f(y_1, \dots, y_N)$ is *shifted symmetric* if it becomes symmetric in new variables $x_i := y_i + N - i$. Let $\text{Sym}^*(N)$ denote the algebra of shifted symmetric polynomials in N variables and let $\text{Sym}(N) \rightarrow \text{Sym}^*(N)$ be the isomorphism determined by the change of variables $x \rightarrow y$. The image in $\text{Sym}^*(N)$ of $S_{\nu|N}^f$ under this isomorphism is called the *shifted Schur polynomial* with index ν and is denoted by $S_{\nu|N}^*$:

$$S_{\nu|N}^*(y_1, \dots, y_N) = S_{\nu|N}^f(y_1 + N - 1, y_2 + N - 2, \dots, y_N).$$

Under the passage from factorial to shifted Schur polynomials the conventional symmetry is lost but stability is recovered: the shifted Schur polynomials enjoy exactly the same property as the ordinary ones, that is

$$S_{\nu|N}^*|_{y_N=0} = S_{\nu|N-1}^*,$$

see [OO97a].

Now we proceed in analogy with the definition of the algebra Sym : Consider the projective limit of the algebras $\text{Sym}^*(N)$ taken with respect to projections $\text{Sym}^*(N) \rightarrow \text{Sym}^*(N-1)$ defined by specializing the last variable to 0, and then take the subalgebra Sym^* formed by elements of bounded degree. The fact that projections $\text{Sym}^*(N) \rightarrow \text{Sym}^*(N-1)$ respect the filtration by the conventional degree of polynomials allows one to equip Sym^* with a structure of filtered algebra. Note that the associated graded algebra gr Sym^* is canonically isomorphic to Sym .

The algebra Sym^* is called the *algebra of shifted symmetric functions* [OO97a]. It is freely generated by 1 and elements

$$p_k^*(y_1, y_2, \dots) = \sum_{i=1}^{\infty} [(y_i - i + \frac{1}{2})^k - (-i + \frac{1}{2})^k], \quad k = 1, 2, \dots$$

Further, the stability property of the N -variate shifted Schur polynomials $S_{\nu|N}^*$ allows one to define their limits as $N \rightarrow \infty$. These are certain elements $S_{\nu}^* \in \text{Sym}^*$ called the *shifted Schur functions* [OO97a]. In the next subsection we explain how to convert them to ordinary symmetric functions.

4.6. The algebra \mathbb{A} of polynomial functions on \mathbb{Y} . For more details about the material of this subsection, see [ORV03] and [IO03].

Definition 4.15. We will need the notion of *modified Frobenius coordinates* of a diagram $\lambda \in \mathbb{Y}$. This is a double collection $(a; b) = (a_1, \dots, a_d; b_1, \dots, b_d)$ of half-integers, where d stands for the number of diagonal boxes in λ , $a_i = \lambda_i - i + \frac{1}{2}$ equals the number of boxes in the i th row of λ plus one-half, and b_i is the same quantity for transposed diagram λ' .

For instance, if $\lambda = (3, 2, 2)$ then $(a; b) = (2\frac{1}{2}, \frac{1}{2}; 2\frac{1}{2}, 1\frac{1}{2})$.

Note that

$$\sum_{i=1}^d (a_i + b_i) = |\lambda|.$$

The transposition map $\lambda \mapsto \lambda'$ switches a and b . Thus, in the coordinates $(a; b)$, rows and columns receive equal rights.

The notion of modified Frobenius coordinates has been suggested in [VK81]; it differs from conventional Frobenius coordinates [Ma95] by the additional terms $\frac{1}{2}$, which makes some formulas more symmetric. An important example is the following nice identity.

Lemma 4.16. *For any Young diagram $\lambda = (a; b) \in \mathbb{Y}$ the following identity holds*

$$\prod_{i=1}^{\infty} \frac{u + i - \frac{1}{2}}{u - \lambda_i + i - \frac{1}{2}} = \prod_{i=1}^d \frac{u + b_i}{u - a_i}. \quad (4.8)$$

Note that the product in the left-hand side is actually finite, because $\lambda_i = 0$ for all i large enough, so that the numerator and denominator cancel out.

Proof. See [IO03, Proposition 1.2]. □

Observe that (4.8) is a rational function in u taking value 1 at $u = \infty$. Hence its logarithm is well defined in a neighborhood $u = \infty$. Expanding it into the Taylor series at $u = \infty$ with respect to the variable $1/u$ we get the identity

$$p_k^*(\lambda_1, \lambda_2, \dots) = p_k(a; -b) := \sum_{i=1}^d [a_i^k - (-b_i)^k], \quad k = 1, 2, \dots \quad (4.9)$$

The expression in the right-hand side is the k th *supersymmetric* power sum in variables $a = (a_i)$ and $-b = (-b_i)$. About the notion of supersymmetric functions, see e.g. [Ma95, §I.3, Ex. 23].

Definition 4.17. Let \mathbb{A} be the algebra of functions on \mathbb{Y} generated over \mathbb{R} by 1 and the functions (4.9). Elements of \mathbb{A} are called *polynomial functions* on \mathbb{Y} , see [KO94].

It is readily verified that the functions (4.9) are algebraically independent. This makes it possible to define two algebra isomorphisms,

$$\mathrm{Sym}^* \rightarrow \mathbb{A} \quad \text{and} \quad \mathrm{Sym} \rightarrow \mathbb{A}. \quad (4.10)$$

The first one identifies the generators $p_k^* \in \mathrm{Sym}^*$ with the functions (4.9), while the second one does the same with the generators $p_k \in \mathrm{Sym}$. In words, the isomorphisms $\mathrm{Sym}^* \rightarrow \mathbb{A} \leftarrow \mathrm{Sym}$ mean that shifted symmetric functions in the row coordinates λ_i of Young diagrams $\lambda \in \mathbb{Y}$ are the same as supersymmetric functions in $(a; -b)$, where $a = (a_i)$ and $b = (b_i)$ are the modified Frobenius coordinates of λ .

This induces an isomorphism $\mathrm{Sym}^* \rightarrow \mathrm{Sym}$ sending p_k^* to p_k for every $k = 1, 2, \dots$. This isomorphism seems to be the most natural way to lift the canonical isomorphism $\mathrm{gr} \mathrm{Sym}^* = \mathrm{Sym}$ to an algebra isomorphism $\mathrm{Sym}^* \rightarrow \mathrm{Sym}$.

The image under $\mathrm{Sym}^* \rightarrow \mathrm{Sym}$ of the shifted Schur functions S_ν^* are some elements of Sym called the *Frobenius–Schur functions* FS_ν . They are studied in detail in [ORV03].

The Frobenius–Schur functions have many nice properties. Here we only note the following one, which directly follows from the very definition:

$$FS_\nu = S_\nu + \text{lower degree terms.}$$

This implies, in particular, that the elements FS_ν with index ν ranging over \mathbb{Y} form a basis in Sym .

Let us agree to identify elements $F \in \mathrm{Sym}$ with the corresponding elements of \mathbb{A} and write them as functions $F(\lambda)$, where the argument λ ranges over \mathbb{Y} . In particular, we will employ the functions $FS_\nu(\lambda)$.

Given $\lambda \in \mathbb{Y}(N)$, set

$$x_i = \lambda_i + N - i, \quad 1 \leq i \leq N. \quad (4.11)$$

The correspondence

$$\lambda \mapsto (x_1, \dots, x_N)$$

is a bijection between $\mathbb{Y}(N)$ and the set

$$\mathbb{Z}_{+, \text{ord}}^N := \mathbb{R}_{\text{ord}}^N \cap \mathbb{Z}_+^N = \{(l_1, \dots, l_N) \in \mathbb{Z}_+^N : l_1 > \dots > l_N\}.$$

Definition 4.18. For every $N = 1, 2, \dots$, define an algebra morphism

$$\pi'_N : \mathrm{Sym} \rightarrow \mathrm{Sym}(N)$$

by setting

$$(\pi'_N(p_k))(x_1, \dots, x_N) = \sum_{i=1}^N [(x_i - N + \tfrac{1}{2})^k - (-i + \tfrac{1}{2})^k], \quad k = 1, 2, \dots$$

Since the right-hand side is a symmetric polynomial, the definition makes sense.

The motivation of this definition is the following: For any $F \in \text{Sym}$ and any $\lambda \in \mathbb{Y}(N) \subset \mathbb{Y}$ one has

$$F(\lambda) = \pi'_N(F)(x_1, \dots, x_N), \quad (x_1, \dots, x_N) \leftrightarrow \lambda.$$

In what follows, the maps π'_N replace the truncation maps π_N .

Remark 4.19. One may view the maps π'_N as a *deformation* of the maps π_N . To see this, introduce the morphisms $\pi_N^{(\varepsilon)} : \text{Sym} \rightarrow \text{Sym}(N)$ by setting

$$\begin{aligned} (\pi_N^{(\varepsilon)}(p_k))(x_1, \dots, x_N) &= \varepsilon^k (\pi'_N(p_k))(\varepsilon^{-1}x_1, \dots, \varepsilon^{-1}x_N) \\ &= \sum_{i=1}^N [(x_i + \varepsilon(-N + \tfrac{1}{2}))^k - \varepsilon^k(-i + \tfrac{1}{2})^k]. \end{aligned}$$

For $\varepsilon = 1$ this coincides with π'_N , and in the limit as $\varepsilon \rightarrow 0$ we get π_N .

Now, the following proposition summarizes the discussion in this and preceding subsections:

Proposition 4.20. *Let us identify the algebra Sym of symmetric functions with the algebra \mathbb{A} of polynomial functions on \mathbb{Y} , as explained above.*

(i) *The maps π'_N relate the Frobenius-Schur functions $FS_\nu \in \text{Sym}$ to the factorial Schur polynomials $S_{\nu|N}^f$. Namely,*

$$\pi'_N(FS_\nu) = \begin{cases} S_{\nu|N}^f, & \text{if } \ell(\nu) \leq N \\ 0, & \text{if } \ell(\nu) > N. \end{cases}$$

(ii) *Let $J'_N \subset \text{Sym}$ denote the kernel of π'_N . The Frobenius-Schur functions FS_ν with $\ell(\nu) > N$ form a basis in J'_N , which implies that*

$$\bigcap_{N=1}^{\infty} J'_N = \{0\}.$$

4.7. The Meixner symmetric functions. Now we follow the same line of arguments as in the Laguerre case.

Definition 4.21 (cf. Definition 4.3). The *Meixner symmetric function* \mathfrak{M}_ν with index $\nu \in \mathbb{Y}$ and complex parameters (z, z', ξ) is given by the following expansion in the Frobenius-Schur symmetric functions:

$$\begin{aligned} \mathfrak{M}_\nu &= \sum_{\mu: \mu \subseteq \nu} (-1)^{|\nu| - |\mu|} \left(\frac{\xi}{1 - \xi} \right)^{|\nu| - |\mu|} \frac{\dim \nu / \mu}{(|\nu| - |\mu|)!} \\ &\quad \times \prod_{\square \in \nu / \mu} (z + c(\square))(z' + c(\square)) \cdot FS_\mu. \end{aligned} \tag{4.12}$$

Like the Laguerre function \mathfrak{L}_ν , the function \mathfrak{M}_ν is an inhomogeneous element of $\text{Sym}_\mathbb{C}$ of degree $|\nu|$, and its top degree homogeneous component coincides with the Schur function S_ν . It follows that the elements \mathfrak{M}_ν form a basis in $\text{Sym}_\mathbb{C}$ consistent with the canonical filtration of $\text{Sym}_\mathbb{C}$ determined by its graduation.

Let

$$\mathbb{C}[z, z', \xi, (1 - \xi)^{-1}] \subset \mathbb{C}(\xi)[z, z']$$

stand for the algebra of polynomials in z and z' whose coefficients are rational functions in ξ with possible poles only at $\xi = 1$. The Meixner symmetric functions may also be viewed as elements of the algebra

$$\text{Sym} \otimes \mathbb{C}[z, z', \xi, (1 - \xi)^{-1}].$$

The above definition is justified by the following proposition, which is parallel to Proposition 4.6:

Proposition 4.22. *For any fixed $\nu \in \mathbb{Y}$, the Meixner symmetric function \mathfrak{M}_ν can be characterized as the only element of the algebra*

$$\text{Sym}[z, z', \xi, (1 - \xi)^{-1}] = \text{Sym} \otimes \mathbb{C}[z, z', \xi, (1 - \xi)^{-1}]$$

such that for any natural N , any $b > 0$, and any $\xi \in (0, 1)$ one has

$$\pi'_N \left(\mathfrak{M}_\nu \Big|_{z=N, z'=N+b-1} \right) = \begin{cases} M_{\nu|N, b, \xi}, & \text{if } \ell(\nu) \leq N \\ 0, & \text{if } \ell(\nu) > N. \end{cases}$$

Proof. The argument is the same as in Proposition 4.6. We use Proposition 4.20 and compare the expansions (4.12) and (3.3). We also use the evident fact that any point set of the form

$$\{(z, z', \xi) = (N, N + b - 1, \xi) : N = N_0, N_0 + 1, \dots, b > 0, \xi \in (0, 1)\} \subset \mathbb{C}^3$$

is a uniqueness set for the algebra $\mathbb{C}[z, z', \xi, (1 - \xi)^{-1}]$: If an element of this algebra vanishes on such a set, then F equals 0. \square

4.8. The Meixner difference operator $\mathfrak{D}^M : \text{Sym}_\mathbb{C} \rightarrow \text{Sym}_\mathbb{C}$.

Definition 4.23 (cf. Definition 4.7). We define the *Meixner operator*

$$\mathfrak{D}^M : \text{Sym}_\mathbb{C} \rightarrow \text{Sym}_\mathbb{C}$$

with complex parameters (z, z', ξ) , $\xi \neq 1$, by its action on the Frobenius–Schur functions:

$$\mathfrak{D}^M FS_\nu = -|\nu| FS_\nu + \frac{\xi}{1 - \xi} \sum_{\square \in \nu^-} (z + c(\square))(z' + c(\square)) FS_{\nu \setminus \square} \quad (4.13)$$

Recall that $\nu^- \subset \nu$ denotes the subset of corner boxes in ν .

Proposition 4.24 (cf. Proposition 4.9). *The Meixner symmetric functions are eigenvectors of the operator \mathfrak{D}^M : we have $\mathfrak{D}^M \mathfrak{M}_\nu = -|\nu| \mathfrak{M}_\nu$ for every $\nu \in \mathbb{Y}$.*

Proof. We apply the principle of analytic continuation of identities. When $z = N$ and $z' = N + b - 1$, the desired equality holds because of (3.6). Since the operator \mathfrak{D}^M depends polynomially on z and z' , we are done. \square

Recall that the Meixner operator $D_N^M : \text{Sym}(N) \rightarrow \text{Sym}(N)$ can be realized as a partial difference operator in N variables (Proposition 3.4). Here is a generalization to the operator $\mathfrak{D}^M : \text{Sym}_{\mathbb{C}} \rightarrow \text{Sym}_{\mathbb{C}}$:

Proposition 4.25. *Upon the identification $\text{Sym} = \mathbb{A}$ the Meixner operator \mathfrak{D}^M is implemented by the following operator in the space of functions on \mathbb{Y} , which can be written in two equivalent forms*

$$\begin{aligned} \mathfrak{D}^M f(\lambda) &= \sum_{\square \in \lambda^+} A(\lambda, \square) f(\lambda \cup \square) + \sum_{\square \in \lambda^-} B(\lambda, \square) f(\lambda \setminus \square) - C(\lambda) f(\lambda) \\ &= \sum_{\square \in \lambda^+} A(\lambda, \square) (f(\lambda \cup \square) - f(\lambda)) + \sum_{\square \in \lambda^-} B(\lambda, \square) (f(\lambda \setminus \square) - f(\lambda)), \end{aligned}$$

where

$$\begin{aligned} A(\lambda, \square) &= \frac{\xi}{1 - \xi} (z + c(\square))(z' + c(\square)) \frac{\dim(\lambda \cup \square)}{(|\lambda| + 1) \dim \lambda}, \quad \square \in \lambda^+, \\ B(\lambda, \square) &= \frac{1}{1 - \xi} \sum_{\square \in \lambda^-} \frac{|\lambda| \dim(\lambda \setminus \square)}{\dim \lambda}, \quad \square \in \lambda^-, \\ C(\lambda) &= \frac{1}{1 - \xi} ((1 + \xi)|\lambda| + \xi z z'). \end{aligned}$$

Proof. As functions of the parameters, the coefficients A , B , and C belong to the algebra $\mathbb{C}[z, z', \xi, (1 - \xi)^{-1}]$. Using the principle of analytic continuation as in Proposition 4.6, this enables us to reduce the claim to the Lemma 4.26 below. In that lemma, we denote by $\mathfrak{D}^{(1)}$ and $\mathfrak{D}^{(2)}$ the above two expressions for the operator \mathfrak{D}^M with parameters (z, z') specialized to $(N, N + b - 1)$. We assume that λ ranges over $\mathbb{Y}(N)$, the subset of Young diagrams with at most N nonzero rows. We also employ the correspondence (4.11) to identify diagrams $\lambda \in \mathbb{Y}(N)$ with the corresponding vectors $(x_1, \dots, x_N) \in \mathbb{Z}_{+, \text{ord}}^N$. \square

Lemma 4.26. *Under the above hypotheses, both $\mathfrak{D}^{(1)}$ and $\mathfrak{D}^{(2)}$ preserve the subspace of functions supported by the subset $\mathbb{Y}(N) \subset \mathbb{Y}$, and their restrictions to $\mathbb{Y}(N)$ coincide with the two equivalent forms of the N -variate Meixner difference operator D_N^M given in Proposition 3.4.*

Proof. If $\lambda \in \mathbb{Y}(N)$ then the diagram $\lambda \cup \square$ can fall outside $\mathbb{Y}(N)$ only if $\ell(\lambda) = N$ and $\square = (N + 1, 1)$, which entails $c(\square) = -N$. However, then $z + c(\square)$ vanishes for $z = N$, which implies vanishing of the coefficient $A(\lambda, \square)$ in front of the term $f(\lambda \cup \square)$. This shows that if $z = N$, then \mathfrak{D}^M can be restricted to $\mathbb{Y}(N)$.

Compare now the above formulas with (3.8) and (3.9), keeping in mind the correspondence $\mathbb{Y}(N) \leftrightarrow \mathbb{Z}_{+, \text{ord}}^N$.

First of all, it is readily verified that $C(x) = C(\lambda)$. Next, observe that the shift $x \rightarrow x + \varepsilon$ amounts to appending a new box \square to the i th row of λ ; likewise the shift $x \rightarrow x - \varepsilon_i$ amounts to removing the last box \square in the i th row. Such an operation is forbidden precisely in the case when $x + \varepsilon_i$ or $x - \varepsilon_i$ falls outside $\mathbb{Z}_{+, \text{ord}}^N$, and then we know that the coefficient $A_i(x)$ or $B_i(x)$ automatically vanishes. Therefore, it remains to check that if appending/removing a box \square in the i th row is possible, then

$$A(\lambda, \square) = A_i(x), \quad B(\lambda, \square) = B_i(x).$$

To see this, one applies the classical Frobenius' dimension formula [Ma95, §I.7, Ex. 6], which in our notation looks as follows

$$\frac{\dim \lambda}{|\lambda|!} = \frac{V_N(x)}{\prod_{i=1}^N x_i!}$$

□

Remark 4.27. The fact that the two expressions given in Proposition 4.25 give the same result is equivalent to the relation

$$\sum_{\square \in \lambda^+} A(\lambda, \square) + \sum_{\square \in \lambda^-} B(\lambda, \square) = C(\lambda) \quad \forall \lambda \in \mathbb{Y}.$$

Excluding from this relation the parameters z , z' , and ξ reduces it to a system of combinatorial identities involving only contents of boxes and ratios of dimensions. These identities first appeared in Kerov's paper [Ke00]; see also [Ol10a, Proposition 5.1].

4.9. The autoduality property of the Meixner symmetric functions. The classical univariate Meixner polynomials are *autodual* in the sense that, in an appropriate standardization, they are symmetric with respect to transposition of the index and the argument, which both range over \mathbb{Z}_+ . The similar autoduality property holds for the Meixner symmetric functions realized as functions on \mathbb{Y} under the identification $\text{Sym} = \mathbb{A}$:

Proposition 4.28. *Change the standardization of \mathfrak{M}_ν by setting*

$$\mathfrak{M}_\nu = \left(\frac{\xi}{1 - \xi} \right)^{|\nu|} \frac{\dim \nu}{|\nu|!} \prod_{\square \in \nu} (z + c(\square))(z' + c(\square)) \mathfrak{M}'_\nu,$$

Then the following autoduality relation holds

$$\mathfrak{M}'_\nu(\lambda) = \mathfrak{M}'_\lambda(\nu), \quad \nu, \lambda \in \mathbb{Y}.$$

Proof. There is a simple expression for the functions $FS_\mu(\cdot)$ on \mathbb{Y} :

$$FS_\mu(\lambda) = \begin{cases} \frac{|\lambda|!}{(|\lambda| - |\mu|)!} \frac{\dim \lambda / \mu}{\dim \lambda}, & \text{if } \mu \subseteq \lambda \\ 0, & \text{otherwise,} \end{cases} \quad (4.14)$$

where λ ranges over \mathbb{Y} ; see [ORV03]. Together with (4.12) this gives

$$\begin{aligned} \mathfrak{M}'_\nu(\lambda) = \sum_{\mu \subseteq (\nu \cap \lambda)} (-1)^{|\mu|} \left(\frac{1 - \xi}{\xi} \right)^{|\mu|} & \frac{|\nu|! |\lambda|!}{(|\nu| - |\mu|)! (|\lambda| - |\mu|)!} \\ & \times \frac{\dim \nu / \mu \dim \lambda / \mu}{\dim \nu \dim \lambda} \prod_{\square \in \mu} \frac{1}{(z + c(\square))(z' + c(\square))}. \end{aligned}$$

Clearly, this expression is symmetric under $\nu \leftrightarrow \lambda$. \square

4.10. Limit transition Meixner \rightarrow Laguerre.

Definition 4.29. Introduce the following notation: Let $\varepsilon \in \mathbb{C} \setminus \{0\}$. Denote by $G : \text{Sym} \rightarrow \text{Sym}$ the operator multiplying every homogeneous element by its degree, and let $\varepsilon^{-G} : \text{Sym} \rightarrow \text{Sym}$ be the operator acting in the m th homogeneous component of Sym as multiplication by ε^{-m} , for each $m \in \mathbb{Z}_+$. Next, set

$$FS_\nu^{(\varepsilon)} = \varepsilon^{|\nu|} \varepsilon^{-G} FS_\nu, \quad \nu \in \mathbb{Y}.$$

Finally, set

$$\mathfrak{M}_\nu^{(\varepsilon)} = \varepsilon^{|\nu|} \varepsilon^{-G} \mathfrak{M}_\nu,$$

where “ \mathfrak{M}_ν ” in the right-hand side denotes the Meixner symmetric function with parameters $(z, z') \in \mathbb{C}^2$ and $\xi = 1 - \varepsilon$.

Proposition 4.30 (cf. Proposition 2.1). *If ν and (z, z') are fixed while $\varepsilon \rightarrow 0$, then*

$$FS_\nu^{(\varepsilon)} \rightarrow S_\nu \quad (4.15)$$

and

$$\mathfrak{M}_\nu^{(\varepsilon)} \rightarrow \mathfrak{L}_\nu, \quad (4.16)$$

where convergence holds in the finite-dimensional subspace $\text{Sym}^{\deg \leq |\nu|} \subset \text{Sym}$.

Proof. The limit relation (4.15) is obvious, because the top degree homogeneous component of FS_ν coincides with S_ν .

To prove (4.16), observe that the expansions (4.3) and (4.12) have the form

$$\mathfrak{L}_\nu = \sum_{\mu: \mu \subseteq \nu} c_{\nu\mu} S_\mu, \quad \mathfrak{M}_\nu = \sum_{\mu: \mu \subseteq \nu} \left(\frac{1 - \varepsilon}{\varepsilon} \right)^{|\nu| - |\mu|} c_{\nu\mu} FS_\mu,$$

where the coefficients $c_{\nu\mu}$ do not depend on ξ . The second relation implies

$$\mathfrak{M}_\nu^{(\varepsilon)} = \sum_{\mu: \mu \subseteq \nu} (1 - \varepsilon)^{|\nu| - |\mu|} c_{\nu\mu} FS_\mu^{(\varepsilon)},$$

which reduces (4.16) to (4.15). \square

Set

$$\mathfrak{D}^{M,\varepsilon} = \varepsilon^{-G} \circ \mathfrak{D}^M \circ \varepsilon^G,$$

where “ \mathfrak{D}^M ” in the right-hand side is the Meixner operator with parameters $(z, z') \in \mathbb{C}^2$ and $\xi = 1 - \varepsilon$.

Corollary 4.31 (cf. Remark 2.2 (ii)). *We have*

$$\lim_{\varepsilon \rightarrow 0} \mathfrak{D}^{M,\varepsilon} = \mathfrak{D}^L.$$

The limit has an evident meaning because the operators in question preserve the canonical filtration in $\text{Sym}_{\mathbb{C}}$.

Proof. Recall that for any $\nu \in \mathbb{Y}$

$$\mathfrak{D}^L \mathfrak{L}_{\nu} = -|\nu| \mathfrak{L}_{\nu}, \quad \mathfrak{D}^M \mathfrak{M}_{\nu} = -|\nu| \mathfrak{M}_{\nu}.$$

The last relation implies

$$\mathfrak{D}^{M,\varepsilon} \mathfrak{M}_{\nu}^{(\varepsilon)} = -|\nu| \mathfrak{M}_{\nu}^{(\varepsilon)}.$$

Since $\mathfrak{M}_{\nu}^{(\varepsilon)} \rightarrow \mathfrak{L}_{\nu}$, this concludes the proof.

Alternatively, one may use Propositions 4.9 and 4.24, and formula (4.16). \square

5. ORTHOGONALITY

5.1. Formal moment functionals on Sym . As above, in this subsection, (z, z', ξ) is an arbitrary triple of complex parameters with $\xi \neq 1$.

Definition 5.1. Introduce linear functionals

$$\varphi^L = \varphi_{z,z'}^L : \text{Sym}_{\mathbb{C}} \rightarrow \mathbb{C}, \quad \varphi^M = \varphi_{z,z',\xi}^M : \text{Sym}_{\mathbb{C}} \rightarrow \mathbb{C}$$

by setting

$$\varphi^L(\mathfrak{L}_{\nu}) = \delta_{\nu\emptyset}, \quad \varphi^M(\mathfrak{M}_{\nu}) = \delta_{\nu\emptyset}.$$

for any $\nu \in \mathbb{Y}$. Since $\mathfrak{D}^L \mathfrak{L}_{\nu} = -|\nu| \mathfrak{L}_{\nu}$ and $\mathfrak{D}^M \mathfrak{M}_{\nu} = -|\nu| \mathfrak{M}_{\nu}$, the above definition is equivalent to saying that the functionals vanish on the range of the operators \mathfrak{D}^L and \mathfrak{D}^M , respectively, and equal 1 on the unity element of $\text{Sym}_{\mathbb{C}}$.

Proposition 5.2. *We have*

$$\varphi^L(S_{\nu}) = (z)_{\nu}(z')_{\nu} \frac{\dim \nu}{|\nu|!} \tag{5.1}$$

$$\varphi^M(FS_{\nu}) = \left(\frac{\xi}{1-\xi} \right)^{|\nu|} (z)_{\nu}(z')_{\nu} \frac{\dim \nu}{|\nu|!} \tag{5.2}$$

Here $\delta_{\nu\emptyset}$ is Kronecker's delta, which is equal to 0 for every $\nu \in \mathbb{Y}$ distinct from \emptyset , and to 1 for $\nu = \emptyset$. Recall that $\mathfrak{L}_{\emptyset} = \mathfrak{M}_{\emptyset} = 1$.

Proof. Examine the Laguerre case. Since φ^L vanishes on the range of \mathfrak{D}^L , (4.4) implies

$$\varphi^L(S_\nu) = \frac{1}{|\nu|} \sum_{\square \in \nu^-} (z + c(\square))(z' + c(\square))\varphi^L(S_{\nu \setminus \square}), \quad \nu \neq \emptyset.$$

Iterate and use the fact that, by the very definition of the quantity $\dim \nu$, it equals the number of all possible ways of successive removals of squares from ν ending at \emptyset . This leads to the desired formula.

The same argument works in the Meixner case; here we use (4.13). \square

Consider the structure constants for multiplication of symmetric functions in the bases $\{\mathfrak{L}_\nu\}$ and $\{\mathfrak{M}_\nu\}$:

$$\mathfrak{L}_\mu \mathfrak{L}_\nu = \sum_\lambda c_{\lambda\mu\nu}^L \mathfrak{L}_\lambda, \quad \mathfrak{M}_\mu \mathfrak{M}_\nu = \sum_\lambda c_{\lambda\mu\nu}^M \mathfrak{M}_\lambda.$$

For fixed (λ, μ, ν) , the structure constants are functions in the parameters:

$$c_{\lambda\mu\nu}^L = c_{\lambda\mu\nu}^L(z, z'), \quad c_{\lambda\mu\nu}^M = c_{\lambda\mu\nu}^M(z, z', \xi).$$

The next lemma is used in the theorem below.

Lemma 5.3. *The Laguerre and Meixner structure constants $c_{\lambda\mu\nu}^L(z, z')$ and $c_{\lambda\mu\nu}^M(z, z', \xi)$ belong to the algebras $\mathbb{C}[z, z']$ and $\mathbb{C}[z, z', \xi, (1 - \xi)^{-1}]$, respectively.*

Proof. By (4.3), the transition matrix between the bases of the Laguerre symmetric functions and the Schur functions is unitriangular with respect to the natural partial order on the index set \mathbb{Y} given by inclusion of Young diagrams, and the matrix entries are polynomials in (z, z') . Therefore, the same holds for the inverse matrix. This shows that the Laguerre structure constants are polynomial in (z, z') .

The same argument works for the Meixner structure constants as well; here we use (4.12). \square

Note that top degree structure constants (that is, those with $|\lambda| = |\mu| + |\nu|$) do not depend on the parameters and coincide with the the Littlewood-Richardson coefficients, which are the structure constants in the basis of Schur functions.

Theorem 5.4. *For any $\mu, \nu \in \mathbb{Y}$,*

$$\varphi^L(\mathfrak{L}_\mu \mathfrak{L}_\nu) = \delta_{\mu\nu} \cdot (z)_\nu (z')_\nu, \tag{5.3}$$

$$\varphi^M(\mathfrak{M}_\mu \mathfrak{M}_\nu) = \delta_{\mu\nu} \cdot \frac{\xi^{|\nu|}}{(1 - \xi)^{2|\nu|}} (z)_\nu (z')_\nu. \tag{5.4}$$

Proof. By the very definition of the linear functionals φ^L and φ^M ,

$$\varphi^L(\mathfrak{L}_\mu \mathfrak{L}_\nu) = c_{\emptyset\mu\nu}^L, \quad \varphi^M(\mathfrak{M}_\mu \mathfrak{M}_\nu) = c_{\emptyset\mu\nu}^M.$$

Thus (5.3) and (5.4) are equivalent to

$$c_{\emptyset\mu\nu}^L = \delta_{\mu\nu} \cdot (z)_\nu (z')_\nu, \quad c_{\emptyset\mu\nu}^M = \delta_{\mu\nu} \cdot \frac{\xi^{|\nu|}}{(1-\xi)^{2|\nu|}} (z)_\nu (z')_\nu.$$

By Lemma 5.3, the left-hand sides of the equalities in questions are elements of the algebras $\mathbb{C}[z, z']$ and $\mathbb{C}[z, z', \xi, (1-\xi)^{-1}]$, respectively. Obviously, the same holds for the right-hand sides as well. Applying the principle of analytic continuation, we reduce the problem to the case when $(z, z') = (N, N + b - 1)$ and, in the Meixner case, $\xi \in (0, 1)$. But then our equalities are reduced to the orthogonality relations for the N -variate Laguerre and Meixner polynomials, see (3.16) and (3.17). \square

Note that the similar functionals on $\text{Sym}(N)$ coincide with expectation with respect to the Laguerre or Meixner version of the weight measure w_N (see (3.15)). In particular, in the case $N = 1$ these are expectations with respect to the classical Laguerre or Meixner weight measure. In the classical theory of orthogonal polynomials, such functionals are called the moment functionals. For this reason, we will call φ^L and φ^M the (formal) *moment functionals* associated with the Laguerre and Meixner symmetric functions, respectively.

5.2. Thoma's simplex and Thoma's cone.

Definition 5.5. (i) The *Thoma simplex* is the subspace Ω of the infinite product space $\mathbb{R}_+^\infty \times \mathbb{R}_+^\infty$ formed by all couples $\omega = (\alpha, \beta)$, where $\alpha = (\alpha_i)$ and $\beta = (\beta_i)$ are two infinite sequences such that

$$\alpha_1 \geq \alpha_2 \geq \cdots \geq 0, \quad \beta_1 \geq \beta_2 \geq \cdots \geq 0 \quad (5.5)$$

and

$$\sum_{i=1}^{\infty} \alpha_i + \sum_{i=1}^{\infty} \beta_i \leq 1. \quad (5.6)$$

We equip Ω with the product topology induced from $\mathbb{R}_+^\infty \times \mathbb{R}_+^\infty$. Note that in this topology, Ω is a compact metrizable space.

(ii) The *Thoma cone* $\tilde{\Omega}$ is the subspace of the infinite product space $\mathbb{R}_+^\infty \times \mathbb{R}_+^\infty \times \mathbb{R}_+$ formed by all triples $\tilde{\omega} = (\alpha, \beta, r)$, where $\alpha = (\alpha_i)$ and $\beta = (\beta_i)$ are two infinite sequences and r is a nonnegative real number, such that (α, β) satisfies (5.5) and the following modification of the inequality (5.6)

$$\sum_{i=1}^{\infty} \alpha_i + \sum_{i=1}^{\infty} \beta_i \leq r.$$

Note that $\tilde{\Omega}$ is a locally compact space in the product topology induced from $\mathbb{R}_+^\infty \times \mathbb{R}_+^\infty \times \mathbb{R}_+$.

We will identify Ω with the subset of $\tilde{\Omega}$ formed by triples $\tilde{\omega} = (\alpha, \beta, r)$ with $r = 1$. The name “Thoma cone” given to $\tilde{\Omega}$ is justified by the fact that $\tilde{\Omega}$ may be viewed as the cone with the base Ω : the ray of the cone passing through a base point $\omega = (\alpha, \beta) \in \Omega$ consists of the triples $\tilde{\omega} = (r\alpha, r\beta, r)$, $r \geq 0$.

We are going to realize the algebra Sym as an algebra of functions on $\tilde{\Omega}$. Since Sym is freely generated by the elements p_1, p_2, \dots , it suffices to assign to every element p_k a function $p_k(\tilde{\omega}) = p_k(\alpha, \beta, r)$ on $\tilde{\Omega}$. This is done as follows:

$$p_k(\alpha, \beta, r) = \sum_{i=1}^{\infty} \alpha_i^k + (-1)^{k-1} \sum_{i=1}^{\infty} \beta_i^k, \quad k = 2, 3, \dots$$

(these are the super power sums in variables (α_i) and $(-\beta_i)$, see [Ma95, §I.3, Ex. 23]) and

$$p_1(\alpha, \beta, r) = r.$$

Proposition 5.6. *The functions p_1, p_2, \dots on $\tilde{\Omega}$ defined in this way are continuous and algebraically independent.*

Proof. Because $\sum \alpha_i \leq r$ and $\alpha_1 \geq \alpha_2 \geq \dots \geq 0$, we have the bound $\alpha_i \leq r/i$ and, likewise, $\beta_i \leq r/i$. Therefore, for any $k \geq 2$, the series $\sum \alpha_i^k$ and $\sum \beta_i^k$ converge uniformly provided that r is bounded from above. By the definition of the topology in $\tilde{\Omega}$, it follows that the functions p_k are continuous for $k \geq 2$.

The continuity of $p_1 = r$ is trivial. Note that the special definition of $p_k(\tilde{\omega})$ in the case $k = 1$ can be justified as follows: Consider the subset

$$\tilde{\Omega}^0 := \left\{ (\alpha, \beta, r) \in \tilde{\Omega} : \sum_{i=1}^{\infty} (\alpha_i + \beta_i) = r \right\} \subset \tilde{\Omega},$$

which is a dense subset of type G_δ . The function $\sum \alpha_i + \sum \beta_i$ is not continuous on $\tilde{\Omega}$, but it coincides with the coordinate function r on $\tilde{\Omega}^0$. Therefore, its unique continuous extension to the whole space $\tilde{\Omega}$ also coincides with r .

To prove that the functions p_1, p_2, \dots are algebraically independent, restrict them on the subset of those $\tilde{\omega} = (\alpha, \beta, r) \in \tilde{\Omega}$ for which all beta coordinates are equal to zero, only finitely many of the alpha coordinates are nonzero, and $r = \sum \alpha_i$. On this subset, the p_k ’s turn to the conventional power sums in $\alpha_1, \alpha_2, \dots$, which are well known to be algebraically independent. \square

The proposition provides an embedding of the algebra Sym into the algebra of continuous functions on the Thoma cone. Given an element $F \in \text{Sym}$, we use the notation $F(\tilde{\omega})$ or $F(\alpha, \beta, r)$ for the corresponding function. Vershik and Kerov [VK90] call such functions *extended symmetric functions* in variables $\alpha = (\alpha_i)$, $\beta = (\beta_i)$, and $\gamma := r - \sum (\alpha_i + \beta_i)$. We prefer to call them *polynomial functions on $\tilde{\Omega}$* . We will identify elements $F \in \text{Sym}$ with the corresponding polynomial functions

$F(\tilde{\omega})$ on $\tilde{\Omega}$. Obviously, if $F \in \text{Sym}$ is homogeneous of degree n , then $F(\tilde{\omega})$ is also homogeneous of degree n as a function on the cone.

Definition 5.7. The *Thoma measure* associated with a point $\tilde{\omega} = (\alpha, \beta, r) \in \tilde{\Omega}$ is an atomic measure on \mathbb{R} given by

$$m_{\tilde{\omega}} = \sum \alpha_i \delta_{\alpha_i} + \sum \beta_i \delta_{-\beta_i} + \gamma \delta_0, \quad \gamma = r - \sum \alpha_i - \sum \beta_i \geq 0,$$

where δ_x denotes the Dirac mass at a point $x \in \mathbb{R}$.

Note that $m_{\tilde{\omega}}$ has finite mass equal to r and is compactly supported. Therefore, $m_{\tilde{\omega}}$ is uniquely determined by its moments.

Proposition 5.8. *The k th moment of the Thoma measure $m_{\tilde{\omega}}$ is equal to $p_k(\tilde{\omega})$ for every $k = 1, 2, \dots$.*

Proof. Immediate from the definition of the functions $p_k(\tilde{\omega})$. \square

Here is a corollary. Note that the restriction of a polynomial function $F(\tilde{\omega})$ to the subset $\Omega \subset \tilde{\Omega}$ is a continuous function $F(\omega)$ on Ω . Since Ω is compact, $F(\omega)$ is bounded on Ω . Thus, we get an algebra morphism $\text{Sym} \rightarrow C(\Omega)$, where $C(\Omega)$ is the Banach algebra of real-valued continuous functions with the supremum norm.

Corollary 5.9. *The image of the map $\text{Sym} \rightarrow C(\Omega)$ is a dense subalgebra in $C(\Omega)$.*

Proof. By Proposition 5.8, the functions $F(\omega)$ separate points of the compact space Ω . Then apply the Stone-Weierstrass theorem. \square

The algebra morphism $\text{Sym} \rightarrow C(\Omega)$ is defined on the generators p_1, p_2, \dots by the same formulas as above, only p_1 is mapped to the constant function 1 on Ω . It follows that the kernel of the morphism is the principal ideal in Sym generated by $p_1 - 1$.

Let P be a probability measure on Ω (here and below all measures are tacitly assumed to be Borel measures). Expectation under P determines a linear functional on Sym , which we denote by ψ_P and call the *moment functional* of P :

$$\psi_P(F) = \langle F, P \rangle = \int_{\Omega} F(\omega) P(d\omega), \quad F \in \text{Sym}.$$

By Corollary 5.9, the moment functional determines the initial measure on Ω uniquely.

Proposition 5.10. *A linear functional $\psi : \text{Sym} \rightarrow \mathbb{R}$ is a moment functional of a probability measure P on Ω if and only if ψ is nonnegative on all Schur functions, equals 1 on the constant function 1, and vanishes on the principal ideal in Sym generated by $p_1 - 1$.*

Proof. This result is essentially due to Vershik and Kerov [VK81]. It is equivalent to the special case of Theorem B in [KOO98] corresponding to special value $\theta = 1$ of the Jack parameter (then the Jack symmetric functions turn into the

Schur symmetric functions). The statement of that theorem speaks about “harmonic functions”¹ on \mathbb{Y} instead of moment functionals on Sym , but this is an equivalent language. Indeed, using the formula

$$p_1 S_\nu = \sum_{\kappa \in \mathbb{Y}: \kappa = \nu \cup \square} S_\kappa$$

one sees that the function $\varphi_P(\nu) := \psi_P(S_\nu)$ is “harmonic” if and only if ψ_P vanishes on the principal ideal generated by $p_1 - 1$. \square

The assertion of Proposition 5.10 may be viewed as a generalization of classical Hausdorff’s theorem characterizing probability measures on the closed unit interval $[0, 1]$ in terms of their moment functionals on $\mathbb{R}[x]$. Indeed, embed the closed interval $[0, 1]$ into Ω by making use of the map

$$[0, 1] \ni x \mapsto (\alpha, \beta) = ((x, 0, 0, \dots), (1 - x, 0, 0, \dots)) \in \Omega.$$

In the particular case when P is concentrated on the image of $[0, 1]$, Proposition 5.10 just reduces to Hausdorff’s theorem.

Corollary 5.11. *Every Schur function S_ν is nonnegative on $\tilde{\Omega}$.*

Proof. Let P be the delta measure at a point $\omega \in \Omega$. By Proposition 5.10, $\psi_P(S_\nu) = S_\nu(\omega) \geq 0$. Thus, the function S_ν is nonnegative on $\Omega \subset \tilde{\Omega}$. Since it is a homogeneous function of degree $|\nu|$, it is nonnegative on the whole cone $\tilde{\Omega}$. \square

Let $\tilde{\omega}_0$ denote the vertex of the Thoma cone: all the coordinates of $\tilde{\omega}_0$ are 0. There is a natural bijection between the space $\tilde{\Omega} \setminus \{\tilde{\omega}_0\}$ and the product space $\Omega \times \mathbb{R}_{>0}$:

$$\tilde{\omega} = (\alpha, \beta, r) \leftrightarrow (\omega, r), \quad \omega := (r^{-1}\alpha, r^{-1}\beta). \quad (5.7)$$

Definition 5.12. Given a probability measure P on the Thoma simplex Ω and a parameter $c > 0$, we construct a probability measure \tilde{P}^c on the cone $\tilde{\Omega}$, depending on $c > 0$, as the push-forward of the product measure $P \otimes \gamma_c$ under the correspondence $(\omega, r) \mapsto \tilde{\omega}$ defined in (5.7) (recall that γ_c denotes the gamma distribution with parameter c , see (2.1)). We call \tilde{P}^c the *lifting* of the measure P with the *lifting parameter* c . The initial measure P can be reconstructed from its lifting \tilde{P}^c by taking the push-forward of \tilde{P}^c under the projection $\tilde{\Omega} \setminus \{\tilde{\omega}_0\} \rightarrow \Omega$.

Definition 5.13. A function $F(\tilde{\omega})$ on $\tilde{\Omega}$ will be called *polynomially bounded* if it admits a global bound of the form

$$|F(\alpha, \beta, r)| \leq \text{const}_1 \cdot (1 + r)^{\text{const}_2}$$

with appropriate constants.

¹This terminology, introduced by Vershik and Kerov in [VK90], is unfortunate, since it disagrees with the conventional terminology adopted in potential theory.

Proposition 5.14. *Let P be an arbitrary probability measure on Ω and \tilde{P} be its lifting to $\tilde{\Omega}$ with some parameter $c > 0$.*

- (i) *For any $F \in \text{Sym}$, the corresponding function $F(\tilde{\omega})$ is polynomially bounded.*
- (ii) *Continuous polynomially bounded functions are integrable with respect to any measure of the form \tilde{P}^c .*

Proof. (i) It suffices to check the claim for $f = p_k$, and then it is obvious because $|p_k(\alpha, \beta, r)| \leq r^k$ by the very definition of $p_k(\tilde{\omega})$.

(ii) This immediately follows from the fact that polynomials in r of arbitrary degree are integrable with respect to the gamma distribution. \square

Since Sym is an algebra, the proposition implies that the functions from Sym are not only integrable but are also square integrable. Thus, they are contained in the Hilbert space $L^2(\tilde{\Omega}, \tilde{P}^c)$.

Proposition 5.15. *Let \tilde{P}^c be as above. The functions from Sym form a dense subspace in $L^2(\tilde{\Omega}, \tilde{P}^c)$.*

Proof. Recall that under the bijection (5.7) between $\tilde{\Omega} \setminus \{\tilde{\omega}_0\}$ and $\Omega \times \mathbb{R}_{>0}$, the measure \tilde{P} turns into $P \otimes \gamma_c$. Thus, there is a natural isometry of Hilbert spaces

$$L^2(\tilde{\Omega}, \tilde{P}^c) \rightarrow L^2(\Omega, P) \otimes L^2(\mathbb{R}_+, \gamma_c).$$

Let $\text{Sym}^\circ = \text{Sym} / (p_1 - 1)$ be the quotient of the algebra Sym by the principal ideal generated by $p_1 - 1$. Given $F \in \text{Sym}$ we denote its image in Sym° by F° . Clearly, $p_1^\circ = 1$ and Λ° is freely generated (as a unital commutative algebra) by $p_2^\circ, p_3^\circ, \dots$

The algebra morphism $\text{Sym} \rightarrow C(\Omega)$ defined just before Corollary 5.9 factors through the quotient algebra Sym° and determines an embedding $\text{Sym}^\circ \rightarrow C(\Omega)$. Under this embedding, the generators $p_2^\circ, p_3^\circ, \dots$ are converted to the functions

$$p_k^\circ(\omega) = \sum_i \alpha_i^k + (-1)^{k-1} \sum_j \beta_j^k, \quad \omega = (\alpha, \beta) \in \Omega, \quad k = 2, 3, \dots$$

By Corollary 5.9, the functions on Ω coming from Sym° are dense in $C(\Omega)$. Therefore, they are also dense in $L^2(\Omega, P)$.

Thus, it suffices to prove that each function on $\Omega \times \mathbb{R}_{>0}$ of the form

$$f^\circ \otimes g, \quad \text{where } f^\circ \in \text{Sym}^\circ, \quad g \in L^2(\mathbb{R}_{>0}, \gamma_c),$$

can be approximated by functions coming from Sym , in the norm of the Hilbert space $L^2(\Omega, P) \otimes L^2(\mathbb{R}_+, \gamma_c)$.

Without loss of generality we may assume that f° comes from a homogeneous element $f \in \text{Sym}$. Indeed, take first an arbitrary element $f \in \text{Sym}$ whose image in Sym° is f° . Writing f as a sum of homogeneous components and multiplying each component by a suitable power of p_1 we get a homogeneous element. On the other hand, this transformation does not affect f° .

Let us show that $f^\circ \otimes g$ can be approximated by functions from Sym , which have the form fh with h an appropriate polynomial in p_1 .

Let m be the degree of the homogeneous element $f \in \text{Sym}$. The function on $\Omega \times \mathbb{R}_>$ coming from fh has the form

$$(fh)(\omega, r) = f^\circ(\omega)r^mh(r), \quad (\omega, r) \in \Omega \times \mathbb{R}_>,$$

while the value of $f^\circ g$ at the same point is

$$(f^\circ g)(\omega, r) = f^\circ(\omega)g(r).$$

Therefore, we have reduced the problem to the following claim:

Let $m \in \mathbb{Z}_+$ be fixed; then any function $g \in L^2(\mathbb{R}_{>0}, \gamma_c)$ can be approximated by functions of the form $r^mh(r)$, where h is a polynomial.

Let us prove this claim. The squared distance in $L^2(\mathbb{R}_{>0}, \gamma_c)$ between the functions g and $r^mh(r)$ is equal to

$$\begin{aligned} \int_0^\infty |g(r) - r^mh(r)|^2 \gamma_c(dr) &= \int_0^\infty \left| \frac{g(r)}{r^m} - h(r) \right|^2 r^{2m} \gamma_c(dr) \\ &= \text{const} \int_0^\infty \left| \frac{g(r)}{r^m} - h(r) \right|^2 \gamma_{c+2m}(dr). \end{aligned}$$

Since the function $g(r)$ is square-integrable with respect to γ_c , the function $g(r)/r^m$ is square-integrable with respect to γ_{c+2m} . Now, the desired claim follows from the next one:

The space of polynomials is dense in the L^2 space with respect to the gamma-distribution with arbitrary parameter.

Finally, the latter claim is a standard fact and can be checked, e. g., in the following way: The characteristic function of the gamma distribution is analytic near 0; it follows that the corresponding moment problem is determinate (see [Si98, Prop. 1.6]); and this in turn implies the density claim (see [Si98, Prop. 4.15]). \square

Proposition 5.16. *Let P and \tilde{P}^c be as above, $\psi_P : \text{Sym} \rightarrow \mathbb{R}$ be the moment functional for P , and $\psi_{\tilde{P}^c} : \text{Sym} \rightarrow \mathbb{R}$ be the moment functional for \tilde{P}^c defined by*

$$\psi_{\tilde{P}^c}(F) = \int_{\tilde{\Omega}} F(\tilde{\omega}) \tilde{P}^c(d\tilde{\omega}), \quad F \in \text{Sym}.$$

The two functionals are related in the following way: If F is homogeneous of degree n then

$$\psi_{\tilde{P}^c}(F) = (c)_n \psi_P(F). \tag{5.8}$$

Proof. Under the bijection (5.7), F turns into the function

$$F(\omega, r) = F^\circ(\omega)r^n, \quad \omega \in \Omega, \quad r > 0.$$

Therefore,

$$\psi_{\tilde{P}^c}(F) = \int_{\Omega} F^\circ(\omega) P(d\omega) \cdot \int_{\mathbb{R}_{>0}} r^n \gamma_c(dr).$$

The first integral equals $\psi_P(F)$ while the second integral equals $(c)_n$. \square

5.3. The orthogonality measure for Laguerre symmetric functions. Let us say that the couple $(z, z') \in \mathbb{C}^2$ is *admissible* if $z \neq 0$, $z' \neq 0$, and $(z)_\nu (z')_\nu \geq 0$ for all $\nu \in \mathbb{Y}$. Obviously, the set of admissible values is invariant under symmetries $(z, z') \rightarrow (z', z)$ and $(z, z) \rightarrow (-z, -z')$, the latter holds because $(-z)_\nu = (-1)^{|\nu|} (z)_{\nu'}$.

It is not difficult to get an explicit description of the admissible range of the parameters (z, z') , see [BO06c, Proposition 1.2]. One can represent it as the union of the following three subsets or *series*:

- The *principal series* is $\{(z, z') : z' = \bar{z} \in \mathbb{C} \setminus \mathbb{Z}\}$.
- The *complementary series* is $\cup_{k \in \mathbb{Z}} \{(z, z') : k < z, z' < k + 1\}$.
- The *degenerate series* comprises the set

$$\{(z, z') = (N, N + b - 1) : N = 1, 2, \dots; b > 0\}$$

together with its images under the symmetry group $\mathbb{Z}_2 \times \mathbb{Z}_2$.

The reason why the values $z = 0$ and $z' = 0$ are forbidden is that then $(z)_\nu (z')_\nu$ vanishes for all $\nu \neq \emptyset$, which is a trivial case.

If (z, z') is in the principal or complementary series, then $(z)_\nu (z')_\nu$ is strictly positive for all ν . If (z, z') is in the degenerate series, then $(z)_\nu (z')_\nu$ is strictly positive for diagrams ν contained in a horizontal or vertical strip, and vanishes otherwise.

Note that $zz' > 0$ for any admissible couple.

Proposition 5.17. *Assume (z, z') is admissible and write $\varphi_{z, z'}^L$ for the Laguerre moment functional with parameters (z, z') .*

There exists a unique probability measure $P = P_{z, z'}$ on the Thoma simplex Ω such that if $F \in \text{Sym}$ is homogeneous of degree n , then

$$\psi_P(F) = \frac{1}{(zz')_n} \varphi_{z, z'}^L(F). \quad (5.9)$$

According to the definition of φ^L this means that

$$\psi_P(S_\nu) = \frac{(z)_\nu (z')_\nu}{(zz')_{|\nu|}} \frac{\dim \nu}{|\nu|!}, \quad \forall \nu \in \mathbb{Y}.$$

Proof. We have to check that the linear functional ψ_P as defined above satisfies the three conditions stated in Proposition 5.10. The first condition (nonnegativity on the Schur functions) holds by the very definition of admissible parameters. The second condition (normalization) is obvious. The third condition (vanishing on the

principal ideal generated by $p_1 - 1$ is a nontrivial claim, for which there exist a few different proofs, see e.g. [Ol03a, §8], [BO00b]. \square

Let us say that a probability measure \tilde{P} on $\tilde{\Omega}$ is an *orthogonality measure* for the Laguerre symmetric functions with given parameters (z, z') if the formal moment functional $\varphi_{z, z'}^L$ serves as the moment functional for \tilde{P} .

Theorem 5.18. *Let (z, z') be admissible, $P_{z, z'}$ be the probability measure on Ω defined in Proposition 5.17, and $\tilde{P}_{z, z'}$ be its lifting with parameter $c = zz'$.*

(i) *The measure $\tilde{P}_{z, z'}$ is an orthogonality measure for the Laguerre symmetric functions with the same parameters (z, z') .*

(ii) *If (z, z') is in the principal or complementary series, then the Laguerre symmetric functions form an orthogonal basis in the Hilbert space $L^2(\tilde{\Omega}, \tilde{P}_{z, z'})$.*

Proof. This follows from (5.9) and (5.8). \square

5.4. The Schur measures. By a *specialization* of the algebra Sym we mean a multiplicative homomorphism $\psi: \text{Sym} \rightarrow \mathbb{C}$, $\psi(1) = 1$. It is uniquely determined by its values $\psi(p_k)$ on the generators p_k , $k = 1, 2, \dots$. Since the p_k 's are algebraically independent, these values may be chosen arbitrarily in \mathbb{C} . We write $\bar{\psi}$ for the conjugate specialization defined by $\bar{\psi}(p_k) = \overline{\psi(p_k)}$.

We start with a simple technical proposition.

Proposition 5.19. *Assume ψ and ψ' are two specializations of Sym such that for some constants $C > 0$ and $\eta > 0$*

$$|\psi(p_k)| \leq C\eta^k, \quad |\psi'(p_k)| \leq C\eta^k, \quad \forall k = 1, 2, \dots \quad (5.10)$$

Then for any $n = 1, 2, \dots$

$$\sum_{\lambda \in \mathbb{Y}_n} |\psi(S_\lambda) \psi'(S_\lambda)| \leq \eta^{2n} \frac{(C^2)_n}{n!}$$

Proof. By Cauchy's inequality,

$$\sum_{\lambda \in \mathbb{Y}_n} |\psi(S_\lambda) \psi'(S_\lambda)| \leq \frac{1}{2} \left(\sum_{\lambda \in \mathbb{Y}_n} \psi(S_\lambda) \bar{\psi}(S_\lambda) + \sum_{\lambda \in \mathbb{Y}_n} \psi'(S_\lambda) \bar{\psi}'(S_\lambda) \right).$$

Therefore, with no loss of generality we may assume $\psi' = \bar{\psi}$.

Denote by (\cdot, \cdot) the canonical inner product in Sym . The Schur functions form an orthonormal basis and the power sum functions p_ρ (where ρ ranges over the set of partitions) form an orthogonal basis. As well known,

$$(p_\rho, p_\rho) = \prod_{k=1}^{\infty} k^{m_k} m_k!,$$

where m_k stands for the multiplicity of k in the partition ρ .

Now, we have

$$\begin{aligned} \sum_{\lambda \in \mathbb{Y}_n} \psi(S_\lambda) \bar{\psi}(S_\lambda) &= \sum_{|\rho|=n} \frac{\psi(p_\rho) \bar{\psi}(p_\rho)}{(p_\rho, p_\rho)} \\ &\leq \eta^{2n} \sum_{1 \cdot m_1 + 2 \cdot m_2 + \dots = n} \frac{(C^2)^{m_1+m_2+\dots}}{\prod_k k^{m_k} m_k!} = \eta^{2n} \frac{(C^2)_n}{n!}. \end{aligned}$$

Here the last equality follows from the generating series

$$\begin{aligned} \sum_{m_1, m_2, \dots = 0}^{\infty} \frac{(C^2)^{m_1+m_2+\dots} t^{1 \cdot m_1 + 2 \cdot m_2 + \dots}}{\prod_k k^{m_k} m_k!} &= \prod_{k=1}^{\infty} \sum_{m_k=0}^{\infty} \frac{(C^2 t/k)^{m_k}}{m_k!} \\ &= \prod_{k=1}^{\infty} \exp(C^2 t^k/k) = \exp\left(\sum_{k=1}^{\infty} \frac{C^2 t^k}{k}\right) = (1-t)^{-C^2}. \end{aligned}$$

□

Corollary 5.20. *If the constant η in (5.10) satisfies $\eta < 1$, then*

$$\sum_{\lambda \in \mathbb{Y}} |\psi(S_\lambda) \psi'(S_\lambda)| < \infty.$$

Definition 5.21. Let ψ and ψ' be two specializations of the algebra Sym satisfying condition (5.10) with a constant $\eta < 1$. Set

$$Z(\psi, \psi') = \sum_{\lambda \in \mathbb{Y}} \psi(S_\lambda) \psi'(S_\lambda)$$

and assume that $Z(\psi, \psi') \neq 0$. The *Schur measure* [Ok01] corresponding to the couple (ψ, ψ') is the complex measure $P_{\psi, \psi'}^{\text{Schur}}$ on the set \mathbb{Y} with weights

$$P_{\psi, \psi'}^{\text{Schur}}(\lambda) = \frac{\psi(S_\lambda) \psi'(S_\lambda)}{Z(\psi, \psi')}, \quad \lambda \in \mathbb{Y}.$$

Observe that if $\psi(S_\lambda) \psi'(S_\lambda) \geq 0$ for all λ , then automatically $Z(\psi, \psi') \neq 0$ and $P_{\psi, \psi'}^{\text{Schur}}$ is a probability measure.

5.5. The \mathbf{z} -measures. We will deal with a 3-parameter family of measures on \mathbb{Y} , which are a special case of Schur measures.

Definition 5.22. Let z , z' , and ξ be complex parameters, $|\xi| < 1$. The associated complex measure on \mathbb{Y} , called the (mixed) z -measure, is defined by

$$P_{z, z', \xi}(\lambda) = (1 - \xi)^{zz'} (z)_\lambda (z)_\lambda \xi^{|\lambda|} \left(\frac{\dim \lambda}{|\lambda|!} \right)^2, \quad \lambda \in \mathbb{Y}. \quad (5.11)$$

Proposition 5.23. *Let $\psi_{z,\eta}$ denote the specialization defined by $\psi_{z,\eta}(p_k) = z\eta^k$ for all $k = 1, 2, \dots$. The z -measure $P_{z,z',\xi}$ coincides with the Schur measure corresponding to the couple $\psi = \psi_{z,\sqrt{\xi}}$, $\psi' = \psi_{z',\sqrt{\xi}}$, with arbitrary choice of the square root.*

Equivalently, one could take $\psi = \psi_{z,1}$ and $\psi' = \psi_{z',\xi}$.

Proof. The following identity holds

$$\psi_{z,1}(S_\lambda) = (z)_\lambda \frac{\dim \lambda}{|\lambda|!}, \quad z \in \mathbb{C}.$$

Indeed, in the particular case $z = N = 1, 2, \dots$ the left-hand side equals $S_\lambda(1, \dots, 1)$ with N units, and the equality follows from the comparison of the “hook formulas” for $\dim \lambda$ and $S_\lambda(1, \dots, 1)$, see [Ma95, §I.3, Ex. 4]. Then the general case follows from the observation that the both sides are polynomials in z .

The identity implies that

$$\psi_{z,\eta}(S_\lambda) \psi_{z',\eta}(S_\lambda) = (z)_\lambda (z')_\lambda (\eta^2)^{|\lambda|} \left(\frac{\dim \lambda}{|\lambda|!} \right)^2.$$

It remains to show that if $|\eta| < 1$, then

$$Z(\psi_{z,\eta}, \psi_{z',\eta}) = (1 - \eta^2)^{-zz'}.$$

This follows from the identity

$$\sum_{\lambda \in \mathbb{Y}_n} \psi_{z,1}(S_\lambda) \psi_{z',1}(S_\lambda) = \frac{(zz')_n}{n!},$$

which is verified by exactly the same computation as in the proof of Proposition 5.19. \square

Remark 5.24. (i) Assume $zz' \neq 0, -1, -2, \dots$ and consider, for arbitrary $n = 0, 1, 2, \dots$, the complex measure on the finite set \mathbb{Y}_n defined by

$$P_{z,z'}^{(n)}(\lambda) = \frac{(z)_\lambda (z')_\lambda}{(zz')_n} \frac{(\dim \lambda)^2}{n!}, \quad \lambda \in \mathbb{Y}_n.$$

The above computation shows that

$$\sum_{\lambda \in \mathbb{Y}_n} P_{z,z'}^{(n)}(\lambda) = 1.$$

In particular, if the quantities $(z)_\lambda (z')_\lambda$ are nonnegative for all λ , which happens for admissible (z, z') , then $P_{z,z'}^{(n)}$ is a probability measure on \mathbb{Y}_n .

(ii) Obviously,

$$P_{z,z',\xi}(\lambda) = (1 - \xi)^{zz'} \frac{(zz')^{|\lambda|}}{|\lambda|!} \xi^{|\lambda|} \cdot P_{z,z'}^{(|\lambda|)}(\lambda).$$

This means that the (mixed) z -measure $P_{z,z',\xi}$ is obtained by mixing out the measures $P_{z,z'}^{(n)}$ for different n by means of the measure on \mathbb{Z}_+ with the weights

$$n \rightarrow (1 - \xi)^{zz'} \frac{(zz')_n}{n!} \xi^n, \quad n = 0, 1, 2, \dots$$

The later measure is a negative binomial distribution provided that $zz' > 0$ and $\xi \in (0, 1)$. It follows that if (z, z') is admissible and $\xi \in (0, 1)$, then $P_{z,z',\xi}$ is a probability measure on \mathbb{Y} .

(iii) The measures $P_{z,z'}^{(n)}$ are called the (non-mixed) z -measures. They first appeared in [KOV93]; see also [KOV04]. The mixed z -measures $P_{z,z',\xi}$ were introduced in [BO00a] and probably served as a guiding example for the introduction of the general Schur measures in [Ok01]. In a different context, examples of Schur-type measures related to Macdonald polynomials appeared earlier in [Fu97]. For additional information about the z -measures, see [Ol03a], [BO00b], [BO06a], [BO06c], [BO09], [BOk00, Example 3].

Proposition 5.25. *Let $F(\lambda)$ be a function on \mathbb{Y} satisfying a bound of the form*

$$|F(\lambda)| \leq \text{const}(1 + |\lambda|)^m, \quad \forall \lambda \in \mathbb{Y} \quad (5.12)$$

with some $m > 0$. Then the series

$$\sum_{\lambda \in \mathbb{Y}} F(\lambda) P_{z,z',\xi}(\lambda)$$

converges absolutely and uniformly with respect to parameters z, z', ξ provided that z and z' range in a bounded region of \mathbb{C} and ξ ranges in a disc $|\xi| \leq 1 - \varepsilon$ with $\varepsilon > 0$.

Proof. The claim reduces to analysis of convergence of the double series

$$\sum_{n=1}^{\infty} n^m |\xi|^n \sum_{\lambda \in \mathbb{Y}_n} |\psi_{z,1}(S_\lambda) \psi_{z',1}(S_\lambda)|.$$

The computation in Proposition 5.19 shows that the interior sum is bounded by $(C^2)_n/n!$, where $C = \max |z|, |z'|$, and the claim becomes evident. \square

Using the isomorphism $\text{Sym} \rightarrow \mathbb{A}$ (see (4.10)) we will regard elements of Sym as functions on \mathbb{Y} . With this agreement we have

Proposition 5.26 (cf. Proposition 5.15). *Assume that (z, z') is admissible and $0 < \xi < 1$, so that $P_{z,z',\xi}$ is a probability measure on \mathbb{Y} . Then all functions from Sym are square-integrable with respect to $P_{z,z',\xi}$. Moreover, these functions form a dense subspace in the real Hilbert space $\ell^2(\mathbb{Y}, P_{z,z',\xi})$.*

Proof. Denote by $\text{NBin}_{c,\xi}$ the negative binomial distribution on \mathbb{Z}_+ with parameters $c > 0$ and $\xi \in (0, 1)$:

$$\text{NBin}_{c,\xi}(n) = (1 - \xi)^c \frac{(c)_n}{n!} \xi^n, \quad n \in \mathbb{Z}_+.$$

We have already encountered with this distribution as the weight measure for the classical Meixner polynomials, but in the present context, it plays a somewhat different rôle, and we will use a different notation.

Next, let us abbreviate $P = P_{z,z',\xi}$. As is seen from Remark 5.24 (ii), the push-forward of P under the projection $\mathbb{Y} \rightarrow \mathbb{Z}_+$ sending λ to $|\lambda|$ is $\text{NBin}_{c,\xi}$, where $c = zz' > 0$. This is the only property of P that we actually need, so that in the argument below P may be an arbitrary probability measure on \mathbb{Y} with such a property.

The first two steps below are parallel to Proposition 5.14.

Step 1. Every function $F \in \text{Sym}$ satisfies a bound of the form (5.12). Indeed, it suffices to check this for the generators p_k , and then (4.9) gives

$$|p_k(\lambda)| \leq \left(\sum_i (a_i + b_i) \right)^k = |\lambda|^k.$$

Step 2. Since Sym is an algebra, the first claim is equivalent to saying that the functions from Sym are P -integrable. By virtue of step 1 we may assume that $|F(\lambda)|$ grows not faster as $|\lambda|^m$. Then the claim about integrability reduces to the obvious fact that the function $n \rightarrow n^m$ on \mathbb{Z}_+ is summable with respect to the negative binomial distribution.

Step 3. To prove that Sym is dense in $\ell^2(\mathbb{Y}, P)$ it suffices to verify that any given function on \mathbb{Y} with finite support can be approximated, in the metric of $\ell^2(\mathbb{Y}, P)$, by elements from Sym . Take l so large that our function is supported by the set $\mathbb{Y}_{\leq l} = \{\lambda \in \mathbb{Y} \mid |\lambda| \leq l\}$. Let χ_l be the characteristic function of $\mathbb{Y}_{\leq l}$. Using the interpolation property of Frobenius–Schur functions, we can find an element $f \in \text{Sym}$ taking any prescribed values on $\mathbb{Y}_{\leq l}$. Therefore, it suffices to approximate any function of the form $f\chi_l$ with given l and $f \in \text{Sym}$.

Step 4. We will show that $f\chi_l$ can be approximated by elements of the form $f \cdot h(p_1) \in \text{Sym}$, where h are appropriate polynomials in one indeterminate. The idea is to reduce this to a one-dimensional problem.

Denoting by m the degree of f we have a bound

$$|f(\lambda)|^2 \leq \text{const}(1 + |\lambda|^{2m})$$

with an appropriate constant in front. Next, let $\|\cdot\|$ denote the norm in $\ell^2(\mathbb{Y}, P)$, and let $\bar{\chi}_l$ be the characteristic function of the set $\{0, 1, \dots, l\} \subset \mathbb{Z}_+$. We have

$$\begin{aligned} \|f\chi_l - f \cdot h(\cdot)\|^2 &= \|f(\chi_l - h(\cdot))\|^2 \\ &\leq \text{const} \sum_{n \in \mathbb{Z}_+} (1 + n^{2m}) |\bar{\chi}_l(n) - h(n)|^2 \text{NBin}_{c,\xi}(n). \end{aligned} \quad (5.13)$$

We claim that there exists a bound of the form

$$(1 + n^{2m}) \text{NBin}_{c,\xi}(n) \leq \text{const} \text{NBin}_{c+2m,\xi}(n), \quad n \in \mathbb{Z}_+,$$

with another appropriate constant in front. Indeed, this follows from two facts: first, $\text{NBin}_{c,\xi}(n)$ is strictly positive for all $n \in \mathbb{Z}_+$; second, for large n

$$\frac{\text{NBin}_{c+2m,\xi}(n)}{\text{NBin}_{c,\xi}(n)} = \text{const} \frac{\Gamma(c+2m+n)}{\Gamma(c+n)} \sim \text{const} n^{2m}.$$

Therefore, (5.13) is bounded from above by

$$\text{const} \sum_{n \in \mathbb{Z}_+} |\bar{\chi}_l(n) - h(n)|^2 \text{NBin}_{c+2m,\xi}(n)$$

with a new constant factor. Now, choosing appropriate h we can make this expression arbitrarily small, because the polynomials are dense in the space $\ell^2(\mathbb{Z}_+, \text{NBin}_{c+2m})$. The last claim is well known: it can be derived, e.g., from the fact that the Laplace transform of NBin_{c+2m} is well defined in a neighborhood of the origin, cf. the end of proof of Proposition 5.15. \square

5.6. The orthogonality measure for Meixner symmetric functions. As above, we interpret elements of Sym as functions on \mathbb{Y} . Recall that in Definition 5.1 we have introduced the formal moment functional φ^M associated with the Meixner symmetric functions \mathfrak{M}_ν . Now it is convenient to use for it a more detailed notation $\varphi_{z,z',\xi}^M$.

Theorem 5.27 (cf. Theorem 5.18). *Let (z, z') be admissible and $\xi \in (0, 1)$.*

(i) *The measure $P_{z,z',\xi}$ is an orthogonality measure for the Meixner symmetric functions with the same parameters z, z', ξ , meaning that the formal moment functional $\varphi_{z,z',\xi}^M$ on Sym coincides with expectation under the z -measure $P_{z,z',\xi}$:*

$$\varphi_{z,z',\xi}^M(F) = \sum_{\lambda \in \mathbb{Y}} F(\lambda) P_{z,z',\xi}(\lambda), \quad \forall F \in \text{Sym}. \quad (5.14)$$

(ii) *If (z, z') is in the principal or complementary series, then the Meixner symmetric functions form an orthogonal basis in the real weighted Hilbert space $\ell^2(\mathbb{Y}, P_{z,z',\xi})$.*

Proof. (i) It suffices to check (5.14) for $F = FS_\nu$. Fix $\nu \in \mathbb{Y}_n$ and compute the right-hand side.

The measures $P_{z,z'}^{(n)}$ satisfy an important relation, called the *coherency relation*, see e.g. [BO06a, Proposition 1.2]:

$$P_{z,z'}^{(n)}(\nu) = \sum_{\lambda \in \mathbb{Y}_{n+1}: \lambda \supset \nu} \frac{\dim \nu}{\dim \lambda} P_{z,z'}^{(n+1)}(\lambda), \quad \nu \in \mathbb{Y}_n.$$

Iterating this relation we get for any $l \geq n$

$$P_{z,z'}^{(n)}(\nu) = \sum_{\lambda \in \mathbb{Y}_l: \lambda \supset \nu} \frac{\dim \nu \dim(\lambda/\nu)}{\dim \lambda} P_{z,z'}^{(l)}(\lambda), \quad \nu \in \mathbb{Y}_n.$$

Multiplying the both sides by

$$\frac{1}{\dim \nu} (1 - \xi)^{zz'} \frac{(zz')_l}{l!} \xi^l$$

gives

$$\frac{(zz' + n)_{l-n}}{l!} \xi^{l-n} \cdot \frac{n! P_{z,z',\xi}(\nu)}{\dim \nu} = \sum_{\lambda \in \mathbb{Y}_l: \lambda \supseteq \nu} \frac{\dim(\lambda/\nu)}{\dim \lambda} P_{z,z',\xi}(\lambda). \quad (5.15)$$

Recall (see (4.14) above) that for any $\nu \in \mathbb{Y}_n$ and $\lambda \in \mathbb{Y}_l$

$$FS_\nu(\lambda) = \begin{cases} \frac{l!}{(l-n)!} \frac{\dim(\lambda/\nu)}{\dim \lambda}, & \nu \subseteq \lambda, \\ 0, & \text{otherwise} \end{cases}$$

In particular, $FS_\nu(\lambda)$ vanishes unless $l \geq n$ and $\lambda \supseteq \nu$. Combining this with (5.15) we get

$$\sum_{\lambda \in \mathbb{Y}} FS_\nu(\lambda) P_{z,z',\xi}(\lambda) = \frac{n! P_{z,z',\xi}(\nu)}{\dim \nu} \sum_{l \geq n} \frac{(zz' + n)_{l-n}}{(l-n)!} \xi^{l-n}.$$

But the latter sum equals $(1 - \xi)^{-zz'-n}$, so that

$$\begin{aligned} \sum_{\lambda \in \mathbb{Y}} FS_\nu(\lambda) P_{z,z',\xi}(\lambda) &= \frac{n! P_{z,z',\xi}(\nu)}{\dim \nu} (1 - \xi)^{-zz'-n} \\ &= \left(\frac{\xi}{1 - \xi} \right)^n (z)_\nu (z')_\nu \frac{\dim \nu}{n!}. \end{aligned}$$

This agrees with the expression for the left-hand side of (5.14) given in (5.2).

Note that the above argument holds under weaker assumptions on the parameters: it suffices to assume that $(z, z', \xi) \in \mathbb{C}^3$ and $|\xi| < 1$.

(ii) Proposition 5.26 says that the functions from Sym form a dense subspace in the Hilbert space $\ell^2(\mathbb{Y}, P_{z,z',\xi})$. By virtue of (i) and Theorem 5.4, the functions \mathfrak{M}_ν are pairwise orthogonal. Finally, since (z, z') is not in the degenerate series, the expression (5.4) for the squared norm $(\mathfrak{M}_\nu, \mathfrak{M}_\nu)$ is strictly positive for all ν . Therefore, the functions \mathfrak{M}_ν form an orthogonal basis in $\ell^2(\mathbb{Y}, P_{z,z',\xi})$. \square

Note that in the case when (z, z') is in the degenerate series, the squared norm $(\mathfrak{M}_\nu, \mathfrak{M}_\nu)$ vanishes for some ν . This means that the corresponding functions \mathfrak{M}_ν vanish on the support of the measure $P_{z,z',\xi}$, so that these functions produce zero vectors in the Hilbert space $\ell^2(\mathbb{Y}, P_{z,z',\xi})$. But the remaining elements \mathfrak{M}_ν still form an orthogonal basis.

5.7. Limit transition Meixner \rightarrow Laguerre for orthogonality measures. Using Frobenius' coordinates of Young diagrams we define a family of embeddings $\iota_\varepsilon: \mathbb{Y} \rightarrow \tilde{\Omega}$ depending on the parameter $\varepsilon > 0$:

$$\mathbb{Y} \ni \lambda = (a; b) \quad \mapsto \quad \tilde{\omega} = (\alpha, \beta, r) \in \tilde{\Omega},$$

where

$$\begin{aligned} \alpha &= (\varepsilon a_1, \dots, \varepsilon a_d, 0, 0, \dots), \\ \beta &= (\varepsilon b_1, \dots, \varepsilon b_d, 0, 0, \dots), \\ r &= \varepsilon |\lambda|. \end{aligned} \tag{5.16}$$

The image $\mathbb{Y}^{(\varepsilon)} = \iota_\varepsilon(\mathbb{Y})$ is contained in $\tilde{\Omega}^0$ and is a discrete subset of $\tilde{\Omega}$ which becomes more and more dense as $\varepsilon \rightarrow 0$. We regard $\mathbb{Y}^{(\varepsilon)}$ with small ε as a grid approximation of the Thoma cone, just as the lattice $\varepsilon\mathbb{Z}$ forms a grid approximation of the real line \mathbb{R} .

Recall that the Thoma simplex Ω was defined as the compact subset of $\tilde{\Omega}$ consisted of triples (α, β, r) with $r = 1$. For every $n = 1, 2, \dots$, the map $\iota_{1/n}$ determines an embedding $\mathbb{Y}_n \rightarrow \Omega$. As $n \rightarrow \infty$, the finite sets $\iota_{1/n}(\mathbb{Y}_n)$ become more and more dense in Ω ; we regard them as a grid approximation of the Thoma simplex.

Theorem 5.28. *Let (z, z') be admissible and $\xi = 1 - \varepsilon \in (0, 1)$. As $\xi \rightarrow 1$, that is, $\varepsilon \rightarrow 0$, the push-forwards $\iota_\varepsilon(P_{z, z', 1-\varepsilon})$ of the mixed z -measures converge to measure $\tilde{P}_{z, z'}$ on arbitrary continuous, polynomially bounded test functions on $\tilde{\Omega}$.*

This implies, in particular, that $\iota_\varepsilon(P_{z, z', 1-\varepsilon}) \rightarrow \tilde{P}_{z, z'}$ weakly.

Proof. As it will be shown, the result holds in a more general context. Start with an arbitrary probability measure P on the Thoma simplex Ω . Fix an arbitrary $c > 0$ and consider the lifting of P with parameter $c > 0$; this is a probability measure on $\tilde{\Omega}$, which we will denote by \tilde{P} .

On the other hand, P gives rise to a sequence $\{P^{(n)} : n = 0, 1, 2, \dots\}$ of probability measures, where $P^{(n)}$ lives on the finite set $\mathbb{Y}_n \subset \mathbb{Y}$ and is defined by

$$P^{(n)}(\lambda) = \dim \lambda \psi_P(S_\lambda), \quad \lambda \in \mathbb{Y}_n.$$

The fact that this is indeed a probability measure follows from Proposition 5.10. Next, we mix up the measures $P^{(n)}$ by means of the negative binomial distribution $\text{NBin}_{c, 1-\varepsilon}$. The result is a probability measure on \mathbb{Y} , we take its push-forward under the embedding ι_ε , and denote the resulting probability measure on $\tilde{\Omega}$ by \tilde{P}_ε .

We are going to prove that \tilde{P}_ε converges to \tilde{P} on continuous polynomially bounded test functions. In the special case $P = P_{z, z'}$ this is exactly the claim of the theorem, because then $\tilde{P} = \tilde{P}_{z, z'}$, $P^{(n)}$ coincides with $P_{z, z'}^{(n)}$, and \tilde{P}_ε coincides with $\iota_\varepsilon(P_{z, z', 1-\varepsilon})$.

Step 1. Consider the measure $P^{(n)}$ and take its push-forward $\iota_{1/n}(P^{(n)})$, which is a measure on the Thoma simplex Ω . As $n \rightarrow \infty$, $\iota_{1/n}(P^{(n)})$ weakly converges to P . Indeed, this is a particular case of a more general result: see [KOO98].

Step 2. Denote by $\widetilde{\text{NBin}}_{c,1-\varepsilon}$ the push-forward of the negative binomial distribution under the embedding $\mathbb{Z}_+ \rightarrow \mathbb{R}_+$ taking $n \in \mathbb{Z}_+$ to $\varepsilon n \in \mathbb{R}_+$. Let ε goes to 0. We claim that then $\widetilde{\text{NBin}}_{c,1-\varepsilon}$ converges to the gamma distribution

$$\gamma_c(dr) = \frac{1}{\Gamma(c)} r^{c-1} e^{-r} dr, \quad r \in \mathbb{R}_+,$$

on continuous test functions on \mathbb{R}_+ with at most polynomial growth at $+\infty$.

Indeed, for $k = 1, 2, \dots$, the k th factorial moment of $\widetilde{\text{NBin}}_{c,1-\varepsilon}$ is given by the expression

$$\int_0^{+\infty} x(x-1) \dots (x-k+1) \widetilde{\text{NBin}}_{c,1-\varepsilon}(dx) = (c)_k \left(\frac{\xi}{1-\xi} \right)^k,$$

which can be rewritten as

$$\int_0^{+\infty} r(r-\varepsilon) \dots (r-(k-1)\varepsilon) \widetilde{\text{NBin}}_{c,1-\varepsilon}(dr) = (c)_k \xi^k.$$

Therefore, all the moments of $\widetilde{\text{NBin}}_{c,1-\varepsilon}$ exist and converge, as $\varepsilon \rightarrow 0$, to the respective moments of γ_c . Note also that characteristic function of the gamma distribution is analytic near 0, so that γ_c is a unique solution to the corresponding moment problem. This readily implies the desired claim.

Step 3. We proceed to the proof that for any continuous, polynomially bounded function F on $\tilde{\Omega}$

$$\lim_{\varepsilon \rightarrow 0} \int_{\tilde{\Omega}} F(\tilde{\omega}) \tilde{P}_\varepsilon(d\tilde{\omega}) = \int_{\tilde{\Omega}} F(\tilde{\omega}) \tilde{P}(d\tilde{\omega}). \quad (5.17)$$

By virtue of Step 2, it suffices to prove (5.17) in the case when F is compactly supported. Indeed, consider the projection $\tilde{\Omega} \rightarrow \mathbb{R}_+$ taking $\tilde{\omega} = (\alpha, \beta, r)$ to r . Under this projection, the push-forwards of the measures \tilde{P}_ε and \tilde{P} are the $\widetilde{\text{NBin}}_{c,1-\varepsilon}$ and γ_c , respectively. Then the result of Step 2 allows us to neglect the tails of the measures.

Thus, we may assume that F vanishes for r large enough.

Step 4. Again by virtue of Step 2, (5.17) holds true when F depends on r only. Subtracting from F an appropriate function depending on r we may assume that F vanishes at the point $\tilde{\omega}_0$, the vertex of the Thoma cone.

Step 5. Let us identify $\tilde{\Omega} \setminus \{\tilde{\omega}_0\}$ with the product space $\Omega \times \mathbb{R}_{>0}$ by means of the correspondence

$$\tilde{\omega} = (\alpha, \beta, \delta) \leftrightarrow (\omega, r), \quad r := \delta > 0, \quad \omega = \delta^{-1} \tilde{\omega} \in \Omega.$$

As F is compactly supported and $F(\tilde{\omega}_0) = 0$, it can be approximated, in the supremum norm, by linear combinations of the factorized functions of the form $F(\omega, r) = f(\omega)g(r)$, where f is a continuous function on the compact space Ω and $g(r)$ is a continuous function on $\mathbb{R}_{>0}$ vanishing outside a closed interval $[a, b]$ with $0 < a < b$.

Step 6. After these simplifications, the limit relation (5.17) is easily derived from the claims of Steps 1 and 2.

Indeed, given r contained in the support $\varepsilon\mathbb{Z}_+$ of the distribution $\widetilde{\text{NBin}}_{c,1-\varepsilon}$, we define $n(r) \in \mathbb{Z}_+$ from the relation $r = \varepsilon n(r)$. If r is bounded from below, which is our case, then $n(r)$ goes to infinity, uniformly on r , as $\varepsilon \rightarrow 0$. Then we have

$$\int_{\widetilde{\Omega}} F(\widetilde{\omega}) \widetilde{P}_\varepsilon(d\widetilde{\omega}) = \int_a^b \widetilde{\text{NBin}}_{c,1-\varepsilon}(dr) \int_{\Omega} f(\omega) \iota_{1/n(r)}(P^{(n(r))})(d\omega)$$

As ε gets small, the interior integral in the right-hand side becomes close to

$$\int_{\Omega} f(\omega) P(d\omega),$$

uniformly on $r \in [a, b]$ (this follows from Step 1). Together with the result of Step 2 this implies that the right-hand side converges to a product of two integrals:

$$\int_a^b g(r) \gamma_c(dr) \cdot \int_{\Omega} f(\omega) P(d\omega),$$

which coincides with the right-hand side of the desired limit relation, by the very definition of lifting. \square

6. APPENDIX: THE CHARLIER SYMMETRIC FUNCTIONS

Let η_θ denote the Poisson distribution on \mathbb{Z}_+ with parameter $\theta > 0$:

$$\eta_\theta = e^{-\theta} \sum_{x \in \mathbb{Z}_+} \frac{\theta^x}{x!} \delta_x.$$

The *classical Charlier polynomials* with parameter θ are the orthogonal polynomials with the weight measure η_θ . The monic Charlier polynomials are given by

$$C_n(x) = \sum_{m=0}^n (-\theta)^{n-m} \frac{n^{\downarrow m}}{m!} x^{\downarrow m}.$$

These are just the polynomial eigenfunctions of the difference operator

$$D^C f(x) = \theta f(x+1) + x f(x-1) - (\theta + x) f(x).$$

The polynomials $C_n(x)$ can be obtained as the degeneration of the Meixner polynomials $M_n(x)$ in the following limit regime for the Meixner parameters (b, ξ)

$$b \rightarrow +\infty, \quad \xi \rightarrow 0, \quad b\xi \rightarrow \theta.$$

All the definitions and results concerning the symmetric Meixner polynomials and the symmetric Meixner functions extend, with simplifications, to the Charlier case. Below we list the main formulas; they are obtained by degeneration from the corresponding formulas for the Meixner functions as

$$z \rightarrow \infty, \quad z' \rightarrow \infty, \quad \xi \rightarrow 0, \quad zz'\xi \rightarrow \theta.$$

The *Charlier symmetric functions* with parameter θ are given by the following expansion in the Frobenius–Schur symmetric functions, cf. (4.12):

$$\mathfrak{C}_\nu = \sum_{\mu: \mu \subseteq \nu} (-\theta)^{|\nu| - |\mu|} \frac{\dim \nu / \mu}{(|\nu| - |\mu|)!} \cdot FS_\mu.$$

The *Charlier operator* in Sym is a degeneration of the Meixner operator (4.13):

$$\mathfrak{D}^c FS_\nu = -|\nu| FS_\nu + \theta \sum_{\square \in \nu^-} FS_{\nu \setminus \square} \quad (6.1)$$

We have

$$\mathfrak{D}^c \mathfrak{C}_\nu = -|\nu| \mathfrak{C}_\nu.$$

The orthogonality measure for the Charlier symmetric functions is the *poissonized Plancherel measure* on \mathbb{Y} (cf. (5.11)):

$$P_\theta(\lambda) = e^{-\theta} \theta^{|\lambda|} \left(\frac{\dim \lambda}{|\lambda|} \right)^2, \quad \lambda \in \mathbb{Y}.$$

The Charlier functions form an orthogonal basis in the weighted Hilbert space $\ell^2(\mathbb{Y}, P_\theta)$. The squared norm of \mathfrak{C}_ν is given by formula (cf. (5.4))

$$(\mathfrak{C}_\nu, \mathfrak{C}_\nu) = \theta^{|\nu|}.$$

The *moment functional* corresponding to P_θ has the form (cf. (5.2))

$$\varphi^c(FS_\nu) = \theta^{|\nu|} \frac{\dim \nu}{|\nu|!}.$$

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